FORM FACTORS: Notes and Examples

Introduction: A *form factor* is introduced in scattering problems to account for the spatial extent of the scatterer. The probability amplitude for a point-like scatterer is modified by a form factor, which takes into account the spatial extent and shape of the target. The scattering probability is expressed as a product of the probability for a point-like target multiplied by the square of the form factor:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}_{\mathrm{point-like}} |F(q^2)|^2$$

For example, in electron scattering at low energies, the cross section for scattering from a point-like target is given by the Rutherford scattering formula. If the target has a finite spatial extent, the cross section can be divided into two factors, the Rutherford cross section and the form factor squared.

In Coulomb scattering, the particular property of the spatial extent sampled is the charge distribution $\rho(\mathbf{R})$ for the object. The form factor is defined by this integral over the volume of the target:

$$F(q^2) \equiv \frac{1}{Ze} \int e^{\frac{-i\mathbf{q}.\mathbf{R}}{\hbar}} \rho(\mathbf{R}) d\tau$$

This simplifies in the case of distributions with spherical symmetry to:

$$F(q^2) = \frac{4\pi\hbar}{Zeq} \int R\rho(R) \sin\left(\frac{qR}{\hbar}\right) dR$$

[Exercise: *This is easy to prove. Take a spherical coordinate system where the axis is in the direction of* q. *Use the volume element* $r^2 \sin\theta dR d\theta d\phi$. *The angle between* $\mathbf{q}$ *and* $\mathbf{R}$ *is* θ. *Then perform the integration with the substitution* $y = iqR\cos\theta$.]

Remember that $\mathbf{q} = \hbar(\mathbf{k} - \mathbf{k_0})$ is the momentum transferred to the target from the projectile in the scattering process. In elastic scattering, $k = k_0$ and simple geometry around the triangle of vectors will show that $\sin\left(\frac{\theta}{2}\right) = \frac{q}{2k}$ where θ in this case is the scattering angle. The form factor is a function of q, and therefore scattering angle. The inclusion of the form factor then alters the angular distribution due to Rutherford scattering. Measurements of the deviation of the angular distribution from Rutherford scattering can then be used to extract information about the spatial extent and shape of the charge distribution.

Form factors are more general than just in nuclear physics, where measurements give information about charge distributions of nuclei. The form factors measured for electron scattering from nucleons gave some of the first evidence that they were composed of sub-structure i.e. quarks and gluons. Form factors, or structure factors, also arise in the scattering of X-rays from materials and allow the structure of the material to be deduced; DNA is a classic example. Looking back at the mathematical form, form factors are really Fourier transforms. The scattering of light by objects and apertures can be described in a similar way.

An Example: Hard Sphere

Imagine a nucleus has a constant charge density, with a radius a:

$$\rho(R) = \frac{Ze}{4\pi R^3/3} \text{ for } R < a$$

$$= 0 \text{ elsewhere}$$

Then, performing integration by parts, the form factor is given by:

$$F(q^2) = \frac{4\pi\hbar}{Zeq} \int R\rho(R) \sin\left(\frac{qR}{\hbar}\right) dR$$

$$= \frac{3\hbar}{qa^3} \int_0^a R \sin\left(\frac{qR}{\hbar}\right) dR$$

$$= \frac{3\hbar}{qa^3} \left(\left[\frac{R\hbar}{q} \cos\left(\frac{qR}{\hbar}\right) \right]_0^a - \int_0^a \frac{\hbar}{q} \cos\left(\frac{qR}{\hbar}\right) dR \right)$$

$$= \frac{3\hbar}{qa^3} \left(\frac{a\hbar}{q} \cos\left(\frac{qa}{\hbar}\right) - \left[\left(\frac{\hbar}{q}\right)^2 \sin\left(\frac{qR}{\hbar}\right) dR \right]_0^a \right)$$

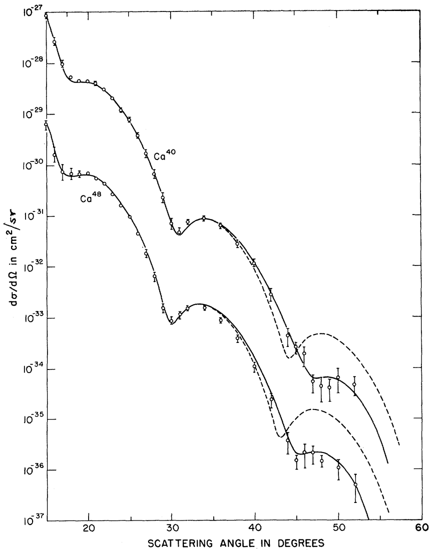
$$= \frac{3\hbar}{qa^3} \left[\frac{a\hbar}{q} \cos\left(\frac{qa}{\hbar}\right) - \left(\frac{\hbar}{q}\right)^2 \sin\left(\frac{qa}{\hbar}\right) \right]$$

$$= 3\left(\frac{\hbar}{qa}\right)^3 \left[\left(\frac{qa}{\hbar}\right) \cos\left(\frac{qa}{\hbar}\right) - \sin\left(\frac{qa}{\hbar}\right) \right]$$

The form factor is zero when the expression in square brackets is zero, i.e. when $\tan(qa/\hbar) = qa/\hbar$. This will be true for various values of q. These correspond to various scattering angles. We would expect a diffraction-like pattern from electron scattering off this target; the scattering probability being this form factor squared multiplied by the Rutherford formula, at least at low energies.

Nuclei do not have sharp edges like this; they have more diffuse surfaces. So good fits to experimental data would not be obtained with this expression. However, form factors from smoothed square distributions do and are used to extract values of charge radius and charge distributions.

Typical Data: The figure below shows data taken from Bellicard et al. Phys. Rev. Letts 19 (1967) 527, showing cross sections for elastic scattering of 757 MeV electrons from ^{40}Ca and ^{48}Ca. Notice that data were taken over several orders of magnitude. The cross section shows diffractive-like patterns with minima as suggested by the example above. Notice that the minima for ^{48}Ca are slightly lower in angle than for ^{40}Ca, due to its lightly larger radius. *Make sure you can explain at least qualitatively the relationship between angle minima and radius; think about the diffraction of light by aperture.*



Other Examples: If you fancy some practise at integrating, you could prove the following:

Exponential charge distribution: Some light nuclei do not have enough protons to build up a constant charge density in their interior and it just falls off from the centre. This can sometimes be described as an exponential:

$$\rho(R) = \frac{eZb^3}{8\pi} e^{-br}$$

where b dictates the fall off. The corresponding form factor is given by:

$$F(q^2) = \left[1 + \frac{q^2}{b^2\hbar^2}\right]^2.$$

Gaussian charge distribution: An alternative for light nuclei is to describe their charge distribution with:

$$\rho(R) = eZ\left(\frac{b^2}{2\pi}\right)^{3/2} e^{-b^2r^2/2}$$

where b dictates the width of the distribution. The corresponding form factor is also exponential:

$$F(q^2) = e^{-q^2/2b^2\hbar^2}.$$

Point-like charge distribution: Using a Dirac delta function:

$$\rho(R) = \frac{\delta(R)eZ}{4\pi}$$

has a form factor of unity, as expected from the definition.