# RUTHERFORD FORMULA FOR POINT-LIKE COULOMBIC SCATTERING

**Orientation**

A point-like charged projectile approaches a charged point-like target along a straight line that would pass by the nucleus at a distance $b$, the impact parameter, if there were no interaction between them. However, the incoming particle is deflected by Coulomb forces such that its outgoing direction, a long way away from the nucleus, is changed by an angle $\theta$, the scattering angle (see diagram below).

**Assumptions**

For now the following assumptions will be made; some can be relaxed as discussed at the end:
(i) Consider only $elastic\ scattering$ where neither the projectile nor the target is raised into an excited state by the collision.
(ii) Consider only the case where the target is very much heavier than the alpha particle; the CM frame is identical to the LAB frame.
(iii) Consider only electromagnetic interactions between the projectile and target using the Coulomb force.
(iv) Analyse using classical dynamics.
(v) Assume that the projectile only scatters once as it transits the thickness of the target material.

Before the collision, a long way from the target, the projectile has negligible electrostatic potential energy and therefore its total energy is purely kinetic $E = \frac{1}{2}mv_0^2$. It has an angular momentum $|\mathbf{r} \times m\mathbf{v}| = mv_0 b$ relative to the target.

During its approach to the target, it reaches a minimum separation distance $r_{min}$, which depends on $b$. The smallest value of $r_{min}$, the distance of closest approach $d$, is for a head-on collision with $b = 0$. At this point its energy is purely electrostatic potential energy. By conservation of energy:

$$\frac{1}{2}mv_0^2 = \frac{zZe^2}{4\pi\epsilon_0 d}$$

where $ze$ and $Ze$ are the charges of projectile and target. Thus

$$d = \frac{zZe^2}{4\pi\epsilon_0 E} = \frac{zZe^2}{2\pi\epsilon_0 mv_0^2}$$

At intermediate points along the trajectory, the energy is made up of contributions from the kinetic energy $\frac{1}{2}mv^2$ and potential energy $\frac{zZe^2}{4\pi r}$, where $r$ is the separation of target and projectile and $v$ is the projectile speed at the particular point of interest.

**Relationship between Impact Parameter and Scattering Angle**

For elastic collisions, the linear momentum in the collision changes only in direction and not magnitude; far from the target the incident and final momenta are both of magnitude $mv_0$ since the target is much more massive than the projectile and does not recoil from the collision.

The change in momentum in the collision is a vector $\Delta\mathbf{p}$, shown in the diagram, whose magnitude can be deduced by geometry to be

$$\Delta p = 2mv_0 \sin\frac{\theta}{2}.$$

This should be equal to the net impulse of the component of the Coulomb force in the direction of $\Delta\mathbf{p}$:

$$\Delta p = \int dp = \int F_{\Delta p} dt = \frac{zZe^2}{4\pi\epsilon_0}\int \frac{\cos\alpha}{r^2} dt.$$

Here the integral is taken from $t = 0$ to the final position $t = \infty$. At the latter point, $\alpha = \pi/2 - \theta/2$ and at the former, $\alpha = -(\pi/2 - \theta/2)$.

In order to do the integral it is useful to transform to an integral over angle rather than time. The tangential component of the angular momentum about the target at a point along the trajectory is $mr^2 d\alpha/dt$ and by conservation of angular momentum:

$$mv_0 b = mr^2 \frac{d\alpha}{dt}$$

$$\frac{dt}{r^2} = \frac{d\alpha}{v_0 b}.$$

The final integral is then:

$$\Delta p = \frac{zZe^2}{4\pi\epsilon_0 v_0 b} \int_{-(\pi/2-\theta/2)}^{+(\pi/2-\theta/2)} \cos\alpha d\alpha = \frac{zZe^2}{4\pi\epsilon_0 v_0 b} 2\cos\frac{\theta}{2}$$

Combining with the original expression for $\Delta p$ gives:

$$b = \frac{d}{2}\cot\frac{\theta}{2}$$

where $d$ is the distance of closest approach deduced above.

**Cross Section**

The scattering has cylindrical symmetry about the beam axis and so the scattering probability is independent of azimuthal angle $\phi$. Taking an annular geometry, where particles incident at a range of impact parameters, from $b$ to $b + db$, forming an annular ring around the target, are scattered into a range of angles , from $\theta$ to $\theta + d\theta$. The area presented by this ring of radius $b$ and thickness $db$ is $2\pi b db$. If there is an incident flux of $\Phi$ projectiles per second per unit area of a plane perpendicular to the beam axis, then the intensity of projectiles with impact parameters from $b$ to $b + db$ is $\Phi 2\pi b db$. Using the expression above for the relationship between $b$, $d$ and $\theta$ this can be written as:

$$dR = \Phi\pi d^2 \frac{\cos\frac{\theta}{2}}{4\sin^3\frac{\theta}{2}} d\theta.$$

All these particles are scattered into a range of angles from $\theta$ to $\theta + d\theta$, which presents a solid angle of $d\Omega = 2\pi\sin\theta d\theta$. The differential cross section is the scattering probability per unit incident flux per solid angle for one target, hence:

$$\frac{d\sigma}{d\Omega} = \frac{1}{\Phi}\frac{dR}{d\Omega} = d^2 \frac{\cos\frac{\theta}{2}}{4\sin^3\frac{\theta}{2}} d\theta \frac{1}{2\sin\theta d\theta}$$

$$= \left(\frac{d}{4}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

$$= \left(\frac{zZe^2}{4\pi\epsilon_0}\frac{1}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

This is known as the Rutherford scattering formula.

If the target is not massive compared to the projectile, $E$ and $\theta$ are reinterpreted as being measured in the centre-of-mass system and the above formula then gives the cross section in that centre-of-mass frame. In this case, the laboratory beam energy is related to $E$ via

$$E = E_{\mathrm{LAB}} \times \frac{m_t}{m_p + m_t}$$

**Discussion**

At very small angles the Rutherford formula suggests that the cross section becomes infinite. This turns out not to be a realistic problem. Firstly, it is difficult to measure at very small angles due to the physical width of any real beam. Secondly, small angles correspond to large impact parameters. When the impact parameter gets comparable to atomic radii, atomic electrons shield the nuclear charge from the projectile and the cross section departs from the Rutherford prediction.

The derivation above uses classical mechanics, but it turns out to be the same in the case of non-relativistic quantum mechanics by mathematical chance! It does need revising in relativistic conditions.

In reality, a projectile could scatter multiple times within the target thickness. If the mean angle per single scatter is $\Theta$, then a random-walk analysis suggests that after $n$ multiple scatters the total angular deviation would be $\Theta\sqrt{n}$. This suggests that the amount of multiple scattering at a particular angle will increase according to the square root of the target thickness, whereas single scattering varies linearly because the chance of scattering is directly proportional to the number of nuclei available to scatter from. This allows an experimental distinction between the two which indicates that plural or multiple scattering only effects the angular distribution at small angles to the beam axis.