ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Applications of Quantum Physics

January 1066, 2.00 p.m. - 3.30 p.m.

Answer **ALL** parts of Question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

You may find the following information useful for this paper.

The Pauli spin matrices are given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenstates of $\widehat{S}_z$ with eigenvalues $+\hbar/2$ and $-\hbar/2$ are denoted by:

$$\alpha_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \beta_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ respectively.}$$

The Landé g-factor is given by:

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}.$$

The Bohr magneton is $\mu_B = e\hbar/2m_e = 5.8 \times 10^{-5}$ eV/T.

1. a) Estimate the probability that a muon with a rest mass of 105.7 MeV/c^2 will penetrate a square barrier that is 5 fm wide, if it has an energy that is 30 MeV below the top of the barrier.

[5 marks]

b) For a spin-$\frac{1}{2}$ particle, write down the spin operators $\widehat{S}_x$, $\widehat{S}_y$ and $\widehat{S}_z$ in terms of Pauli spin matrices and hence verify that:

$$[\widehat{S}_x, \widehat{S}_z] = -i\hbar\widehat{S}_y.$$

[5 marks]

c) A hydrogen atom is in a state described by the wave function:

$$\phi = N(2\psi_{100} + 3\psi_{211} + \psi_{21-1}),$$

where ψ_{nlm} are normalised energy eigenfunctions labelled with the usual quantum numbers.

Find the normalisation constant N. Calculate the expectation value of the z component of angular momentum.

[9 marks]

d) Two distinguishable fermions each have a spin quantum number $s = 3/2$ and are both in the same spatial state. What are the possible values of the quantum numbers S and M_S associated with the total spin of the system?

[4 marks]

Explain briefly how the answer changes if the particles are identical.

[2 marks]

2. a) A particle of mass m moves in a potential given by:

$$V(x) = 0 \qquad \text{for } 0 \le x \le a$$
$$= \infty \qquad \text{elsewhere.}$$

Write down the eigenvalues and normalised eigenfunctions for this system, clearly indicating the allowed values of any quantum numbers involved. How do the energy levels compare to those of a finite square well? Give a brief qualitative explanation for your answer.

[7 marks]

What are the energy levels of a cuboidal quantum dot with sides of length a, a and $a/2$? Give a brief justification for your answer. Deduce the first three magic numbers associated with loading the quantum dot with fermions that have an intrinsic spin $s = \frac{3}{2}$, clearly explaining how you reach your answer.

[9 marks]

b) A particle moves in a potential well given by:

$$V(x) = V_0 \sin(\pi x/a) \qquad \text{for } 0 \le x \le a$$
$$= \infty \qquad \text{elsewhere.}$$

Use perturbation theory to calculate the first-order difference in energy of the lowest level compared to an infinite square well of the same width. Show that this value is a good estimate if V_0 satisfies

$$V_0 \ll \frac{3}{16}\frac{\pi^3\hbar^2}{ma^2}.$$

[9 marks]

3. a) Explain what is meant by the term *g factor*.

[2 marks]

b) An electron is stationary in a constant magnetic field of strength B, which is oriented along the z axis. At $t = 0$, the particle is in a state described by the eigenfunction corresponding to $s_x = +\frac{1}{2}$,

$$\alpha_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find the probability that the particle will be found at a later time t with $s_y = +\frac{1}{2}$.

[9 marks]

You may use:

$$\alpha_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

c) Explain what is meant by the Zeeman effect when an LS-coupled atom is placed in a weak constant magnetic field. Specify what is meant by a weak field in this context.

[7 marks]

A particular LS-coupled atomic state splits into three Zeeman components separated by 0.087 meV when placed in a magnetic field of 1 T. What is the total angular momentum quantum number J of this state? Find the associated Landé g factor. Use your answers to determine which of the following combinations of orbital L and spin S angular-momentum quantum numbers are appropriate for this state, given that both L and S are less than or equal to 1.

[7 marks]

4. a) Explain how information may be encoded using a two-state quantum system. Why is such quantum information different from classical information? Give an example of a two-state system and describe a physical process that might be used to perform operations on its qbits.

[6 marks]

b) The Hadamard operator is often used in quantum logic gates and is defined as:

$$\widehat{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Find the effect of operating with $\widehat{X}$ on the single-particle spin eigenstates, α_z and β_z. What would the outcomes be if measurements of $\widehat{S}_x$ were made on the resulting states? What is the effect of applying the Hadamard operator twice?

[6 marks]

You may use:

$$\alpha_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \beta_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

c) A CNOT operator $\widehat{C}$ acts on a pair of spin-$\frac{1}{2}$ qbits. If the first qbit is in the state α_z, the operator flips the spin state of the second qbit. However, if the first qbit is in the state β_z, there is no effect.

Perform the operation $\widehat{C}\widehat{X}_1$ on pair of qbits in the state $\alpha_z(1)\alpha_z(2)$ where $\widehat{X}_1$ only acts on the first qbit. Explain why the result is an entangled state.

[7 marks]

d) A device is built which takes as an input a pair of spin-$\frac{1}{2}$ qbits. If the two qbits are in spin eigenstates, α_z or β_z, the device copies the state of the first qbit to the second qbit. Show that this device fails to copy the state of the first qbit, if this is a general superposition of spin eigenstates.

[6 marks]

END OF EXAMINATION PAPER