ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Applications of Quantum Physics

January 2017, 2.00 p.m. - 3.30 p.m.

Answer **ALL** parts of Question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

You may find the following information useful for this paper.

The Pauli spin matrices are given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenstates of $\widehat{S}_z$ with eigenvalues $+\hbar/2$ and $-\hbar/2$ are denoted by:

$$\alpha_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \beta_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ respectively.}$$

The Landé g-factor is given by:

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}.$$

The Bohr magneton is $\mu_B = e\hbar/2m_e = 5.8 \times 10^{-5}$ eV/T.

1. a) Briefly explain what is meant by the terms: (i) quantum dot, (ii) quantum wire and (iii) quantum well.

[4 marks]

b) With reference to spin-orbit coupling, briefly explain why the transition from the $3p$ to $3s$ states of the outermost electron in a sodium atom results in a pair of spectral lines.

[5 marks]

c) Express $\widehat{J}_y$ in terms of the ladder operators $\widehat{J}_\pm = \widehat{J}_x \pm i\widehat{J}_y$ and hence show that the expectation value of $\widehat{J}_y$ for a particle in an eigenstate of $\widehat{J}_z$ is zero.

[5 marks]

d) Give an example of a two-state quantum-mechanical system and explain how it might be used as a q-bit. Describe a physical process that might be used to perform operations on the q-bits in your example.

[5 marks]

e) An electron sits in a magnetic field B that points in the $+z$ direction.

i) Write down the Hamiltonian describing the interaction of the electron spin with the field, in terms of $\widehat{S}_z$. What are the associated energies?

[2 marks]

ii) At $t = 0$, the electron is in the state

$$\gamma = \frac{1}{\sqrt{17}} \begin{pmatrix} 1 \\ 4i \end{pmatrix}.$$

Find an expression for the state of the electron as a function of time.

[4 marks]

2. a) The probability that a particle with mass m and energy E tunnels through a one-dimensional potential barrier $V(x)$ can be approximated by:

$$T \simeq \exp\left[-2\int_a^b \sqrt{\frac{2m}{\hbar^2}\Big(V(x)-E\Big)}\ dx\right].$$

Without performing a detailed derivation, explain how this formula arises and explain the circumstances under which it is valid. What do the limits a and b of the integral correspond to?

[8 marks]

b) A particle of mass m is incident on a parabolic barrier of the form $V(x) = V_0 - \alpha x^2$, where α is a constant. The incident energy E is less than V_0, the maximum value of the potential.

Find the values of a and b in this case.

[2 marks]

Show that an estimate for the tunnelling probability is given by:

$$T \simeq \exp\left[-\pi(V_0 - E)\sqrt{\frac{2m}{\alpha\hbar^2}}\right].$$

You may use the standard integral:

$$\int \sqrt{1-z^2}\ dz = \frac{1}{2}\Big(\arcsin(z) + z\sqrt{1-z^2}\Big)\ + C$$

where C is a constant.

[8 marks]

c) A particle is incident on two identical potential barriers. Sketch the form of the transmission probability as a function of incident energy and give a brief qualitative account of the main features.

[7 marks]

3. a) Explain what is meant by the term *magic numbers*. Describe an example of a quantum system that exhibits them, illustrating how the magic numbers are reflected in a physical property of the system.

[5 marks]

b) Electrons are confined in a quantum well by three identical harmonic oscillator potentials in the x, y and z directions. Find the energies and degeneracies associated with the first *three* levels in the well. What magic numbers result from filling these levels with electrons?

[8 marks]

On the basis of the vibrational degeneracies, what is the value of the orbital angular-momentum quantum number ℓ for the first two levels? What combination of ℓ values characterises the third level?

[5 marks]

c) A particle of mass m is located in a perturbed harmonic potential

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \lambda x^4.$$

Use first-order perturbation theory to find the ground-state energy. You may use the information given below. Under what circumstances would your answer be a good estimate?

[7 marks]

The normalised ground-state eigenfunction of a one-dimensional harmonic oscillator is given by:

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}x^2\right].$$

A useful standard integral has the form:

$$\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2}\, dx = \frac{1 \times 3 \times 5 \times .... \times (2n-1)}{2^n}\sqrt{\frac{\pi}{\alpha^{2n+1}}}\quad .$$

4. a) Explain what is meant by quantum entanglement, illustrating your answer with a specific example of an entangled state for a two-particle system.

[4 marks]

b) Using the Pauli spin matrices, verify that:

$$\widehat{S}_+\beta_z = \hbar\alpha_z \quad \text{and} \quad \widehat{S}_-\alpha_z = \hbar\beta_z \ ,$$

where $\widehat{S}_\pm = \widehat{S}_x \pm i\widehat{S}_y$ are the ladder operators for a spin-$\frac{1}{2}$ system.

What are the results of the operations $\widehat{S}_+\alpha_z$ and $\widehat{S}_-\beta_z$?

[5 marks]

c) What is the largest value of the z-component of the *total* spin of two spin-$\frac{1}{2}$ particles? Write down a product of spinors that describes this state.

[2 marks]

For the combined system, the lowering operator is defined as $\widehat{S}_- = \widehat{S}_-^{(1)} + \widehat{S}_-^{(2)}$, where superscripts refer to the two different particles.

By successive operations of $\widehat{S}_-$ on your spinor product, show that there are in total three magnetic substates. What is the quantum number of the total spin for these states?

[7 marks]

d) An electron and positron can form a bound system called positronium, similar to an electron and a proton in the hydrogen atom, where the lowest states have no orbital angular momentum. The degeneracy of the levels formed by different couplings of intrinsic spin ($\widehat{\mathbf{S}} = \widehat{\mathbf{S}}^{(1)} + \widehat{\mathbf{S}}^{(2)}$) is lifted by a spin-dependent interaction of the form:

$$\widehat{H} = \gamma\widehat{\mathbf{S}}^{(1)}.\widehat{\mathbf{S}}^{(2)}.$$

Find the separation in energy of the lowest singlet and triplet states.

[7 marks]

END OF EXAMINATION PAPER