ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Applications of Quantum Physics

1st April 1066, 2.00 p.m. - 3.30 p.m.

Answer **ALL** parts of Question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

You may find the following information useful for this paper.

The Pauli spin matrices are given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenstates of $\widehat{S}_z$ with eigenvalues $+\hbar/2$ and $-\hbar/2$ are denoted by:

$$\alpha_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \beta_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ respectively.}$$

1. a) Estimate the probability of a proton tunnelling through a square barrier with a width of 1 fm, if its energy is 30 MeV below the top of the barrier.

[5 marks]

b) The state of a spin-1/2 particle is described by the normalised spinor

$$\gamma = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

i) If a measurement is made of S_z, what are the possible outcomes and their associated probabilities?

[4 marks]

ii) Calculate the expectation value of a measurement of S_y.

[5 marks]

c) An electron is trapped inside a cuboidal quantum dot with sides of length b, b, and $2b$. Write down an expression for the energy levels of this system, clearly specifying the ranges of the quantum numbers used.

[4 marks]

d) A particle with charge q sits in a one-dimensional infinite square well extending from $x = 0$ to $x = a$. It is subjected to a weak electric potential $\mathcal{E}$, which adds an additional term $\widehat{H}_1 = q\mathcal{E}\left(x - \frac{a}{2}\right)$ to the Hamiltonian. Calculate, to first order, the energy shift of the ground state due to this potential.

[5 marks]

e) State whether the two identical particles in the following entangled wave function are bosons or fermions, giving reasons for your answer:

$$\psi(1,2) = \frac{1}{\sqrt{2}} \left(\psi_n(1)\psi_m(2) + \psi_m(1)\psi_n(2)\right).$$

[2 marks]

2. Briefly describe the implications of the following commutators on what can be determined about the angular momentum of a quantum mechanical system:

$$\left[\widehat{L}_x, \widehat{L}_y\right] = i\hbar \widehat{L}_z$$

$$\left[\widehat{L}^2, \widehat{L}_x\right] = 0,$$

and similar for cyclic permutation of x, y and z.

[3 marks]

What is the result of the action of the operators $\widehat{L}_\pm = \widehat{L}_x \pm i\widehat{L}_y$ on an eigenfunction of $\widehat{L}^2$ and $\widehat{L}_z$ with quantum numbers ℓ and m, respectively?

[2 marks]

Derive an expression for the operator $\widehat{L}_x$ in Cartesian coordinates and write down similar expressions for $\widehat{L}_y$ and $\widehat{L}_z$. Hence show that:

$$\widehat{L}_+ = \hbar z \left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right] - \hbar \left[x + iy\right] \frac{\partial}{\partial z}.$$

[6 marks]

A particle is described by the wave function $\psi = Aze^{-\beta(x^2+y^2+z^2)}$ where A is the normalisation and β is a positive constant.

a) Find the derivatives $\frac{\partial \psi}{\partial x}$, $\frac{\partial \psi}{\partial y}$ and $\frac{\partial \psi}{\partial z}$.

[3 marks]

b) Show that this wave function is an eigenfunction of $\widehat{L}_z$ with quantum number $m = 0$.

[3 marks]

c) Evaluate $\widehat{L}_+\psi$.

[4 marks]

d) A second operation of $\widehat{L}_+$ results in a value of zero. Explain what can be deduced about the value of the quantum number ℓ?

[4 marks]

3. a) Explain briefly the physical origin of the spin-orbit interaction in a one-electron atom.

[4 marks]

The spin-orbit term in the Hamiltonian can be written as $\widehat{H}_{so} = A_L \widehat{\mathbf{L}} \cdot \widehat{\mathbf{S}}$ where A_L is a constant.

A configuration with quantum numbers L and S is split into a series of levels by the spin-orbit interaction. Find an expression for the energetic separation of two states with quantum numbers J and $J-1$.

[8 marks]

b) A magnesium atom has two electrons outside a noble gas core. Fine structure effects produce a group of three low-lying levels with quantum numbers $J = 0$, 1 and 2.

Given that all these levels have a total spin quantum number $S = 1$, deduce the value of the quantum number L.
The excitation energies of the $J = 0$, 1 and 2 levels are 2.7091, 2.7116 and 2.7166 eV respectively. Compare the energy separations of the three magnesium levels with those expected for the spin-orbit interaction and comment on your results.

[5 marks]

c) Write down the additional term that appears in the Hamiltonian when an atom is placed in a *strong* magnetic field oriented along the z axis. Draw a diagram of the magnesium atomic energy levels arising in a strong field from the $S = 1$ states discussed above. Clearly indicate the relative separation of the levels and their degeneracy, and label the levels with all associated quantum numbers.

[8 marks]

4. Show that the following spinors are eigenstates of the operator $\widehat{S}_y$ and determine their eigenvalues:

$$\alpha_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \beta_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Express α_z and β_z in terms of α_y and β_y.

[4 marks]

Explain how information can be encoded using two-level quantum systems. Briefly indicate how this differs from classical information storage.

[4 marks]

Two spies, Amanda and Brenda, want to secretly share information deduced from entangled pairs of spin-$\frac{1}{2}$ particles initially prepared in the following wave function:

$$\psi(A, B) = \frac{1}{\sqrt{2}} \left[\alpha_z(A)\beta_z(B) - \beta_z(A)\alpha_z(B)\right],$$

where A and B label the two particles.

With reference to this wave function, explain what is meant by quantum entanglement.

[4 marks]

Amanda and Brenda each receive one particle from the pair and make a measurement of a spin component using an analyser for S_z or S_y, choosing randomly between the two.

Explain how the results of the spies' measurements compare in the circumstances where: (i) both spies use the S_z analyser; and (ii) one spy uses the S_z analyser and the other S_y, giving probabilities for any particular outcomes under each circumstance.

[4 marks]

Evan, an eavesdropper, intercepts the spin-$\frac{1}{2}$ particles intended for Brenda. Evan makes a measurement, choosing randomly between S_z or S_y analysers, and then retransmits the particle towards Brenda in an eigenstate corresponding to his measurement. Explain how Amanda and Brenda could detect the presence of Evan using the subset of data where they both used the S_z analyser.

[9 marks]

END OF EXAMINATION PAPER