<u>ONE HOUR THIRTY MINUTES</u>

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Applications of Quantum Physics

1st April 1066, 2.00 p.m. - 3.30 p.m.

Answer **<u>ALL</u>** parts of Question 1 and **<u>TWO</u>** other questions

---

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

You may find the following information useful for this paper.

The Pauli spin matrices are given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

α_z and β_z denote the eigenstates of $\widehat{S}_z$ with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively.

The Landé g-factor is given by:

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}.$$

The Bohr magneton is $\mu_B = e\hbar/2m_e = 5.8 \times 10^{-5}$ eV/T.

1. a) Explain briefly what is meant by the quantum numbers S, L and J in atomic physics.

A particular atomic configuration of electrons has the quantum numbers $L = 2$, $S = 1$ and $J = 3$. If the atom is subjected to an external magnetic field of 0.1 T, indicate how many energy levels correspond to this configuration and calculate their separation in energy.

You may assume that the splitting is considerably less than that associated with LS coupling.

[7 marks]

b) What is meant by: (i) a classical computer bit and (ii) a quantum Q-bit. Give a physical example of each. Explain briefly how these two types of bit differ.

[5 marks]

c) Write down the spin operators $\widehat{S}_x$, $\widehat{S}_y$ and $\widehat{S}_z$ using Pauli spin matrices. Hence show that

$$[\widehat{S}_y, \widehat{S}_x] = -i\hbar \widehat{S}_z.$$

[5 marks]

d) A system is composed of two different particles with quantum numbers $j_1 = 5/2$ and $j_2 = 3/2$ describing their angular momenta. What are the allowed values of the quantum numbers J and M_J associated with the total angular momentum of the whole system?

[4 marks]

e) State whether the following two-particle wave functions represent entangled states, giving reasons for your answers.

$$\psi_A = \frac{1}{\sqrt{2}}\Big(\alpha_z(1)\beta_z(2) - \beta_z(1)\beta_z(2)\Big)$$

$$\psi_B = \frac{1}{\sqrt{2}}\Big(\alpha_z(1)\alpha_z(2) + \beta_z(1)\beta_z(2)\Big)$$

[4 marks]

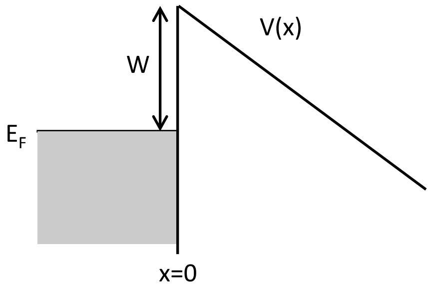
2. a) The probability that a particle with mass m and energy E tunnels through a one-dimensional potential barrier $V(x)$ is approximated by:

$$T \sim \exp\left[-2\int_a^b \sqrt{\frac{2m}{\hbar^2}\big(V(x)-E\big)}\;dx\right].$$

Without performing a detailed derivation, explain how this formula arises. What do the points $x = a$ and $x = b$ correspond to? Under what circumstances is this a good approximation?

[8 marks]

b) The diagram below shows a model of the potential experienced by an electron near a metallic surface at $x = 0$, which is subjected to a constant electric field $\mathcal{E}$. Electronic states within the metal are filled up to the Fermi energy E_F, as indicated on the diagram. The potential in the region $x > 0$ can be taken as $V(x) = E_F + W - e\mathcal{E}x$, where W is the work function of the metal.



Show that an estimate for the tunnelling probability of an electron with energy E_F is given by:

$$T \sim \exp\left[-\frac{4}{3}\sqrt{\frac{2mW^3}{\hbar^2}}\frac{1}{e\mathcal{E}}\right].$$

[10 marks]

c) A tungsten probe on a scanning tunnelling microscope is held at a voltage of 10 V. Estimate the factor by which the current drawn by the probe changes if the distance to the sample is reduced from 5.0 to 4.9 nm. The work function of tungsten can be taken as 4.8 eV.

[7 marks]

3. a) An electron is in a spin state described by $\chi = \frac{1}{\sqrt{2}}(\alpha_z - \beta_z)$. If a measurement is made of the z component of the electron spin, what are the possible values that might be measured and with what probabilities?

[2 marks]

b) Spin ladder operators $\widehat{S}_{\pm} = \widehat{S}_x \pm i\widehat{S}_y$ have the following effect when operating on eigenstates of $\widehat{S}_z$:

$$\widehat{S}_{+}\beta_z = \hbar\alpha_z \quad \text{and} \quad \widehat{S}_{-}\alpha_z = \hbar\beta_z.$$

Without detailed derivation, state the results of the operations $\widehat{S}_{+}\alpha_z$ and $\widehat{S}_{-}\beta_z$. Express $\widehat{S}_x$ and $\widehat{S}_y$ in terms of the ladder operators. Show that χ is an eigenstate of $\widehat{S}_x$ and find the associated eigenvalue.

[7 marks]

c) An electron is subjected to a magnetic field B in the $+z$ direction. Write down the Hamiltonian describing the interaction of the electron spin with the magnetic field, if the electron has no orbital motion in the field.

[2 marks]

The initial spin state of the electron is described by $\psi(0) = \chi$. Using the time-dependent Schrödinger equation, verify that the spin state $\psi(t)$ of the electron as a function of time can be written as:

$$\psi(t) = \frac{1}{\sqrt{2}}(\alpha_z e^{-i\omega t} - \beta_z e^{i\omega t})$$

and thus determine an expression for ω.

[6 marks]

Find the expectation value of the x and y components of the electron spin as a function of time. Use your result to describe qualitatively the motion of the spin.

[8 marks]

4. a) Explain what is meant by the terms: (i) quantum dot, (ii) quantum wire and (iii) quantum well.

[3 marks]

b) A cuboidal quantum dot has sides of length $a/2$ in the x direction and a in the y and z directions. The potential energy of an electron inside the dot is zero, but rises so quickly at the walls that it can be considered infinite elsewhere.

Without performing a detailed derivation, what are the energies of the first three levels for a single electron in the dot? Determine the degeneracies of these levels. What "magic" numbers are expected when filling these levels with electrons?

[9 marks]

Write down the wave function $\psi(x, y, z)$ for the lowest single-electron energy level in the quantum dot using a coordinate system that has an origin at one of the corners. Determine the associated normalisation factor.

[4 marks]

c) This quantum dot is subjected to an additional potential $V = V_0 \sin(\pi z/a)$. Use first-order perturbation theory to calculate the change in the energy of the lowest level. Under what conditions is this a good approximation?

[9 marks]

END OF EXAMINATION PAPER