

ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Applications of Quantum Physics

17th January 2012, 2.00 p.m. - 3.30 p.m.

Answer **ALL** parts of question 1 and **TWO** other questions

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

In this paper α_z and β_z denote the eigenstates of \hat{S}_z with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively.

You may use the standard set of Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

You may also use the Landé g -factor,

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)},$$

and the value for the Bohr magneton, $\mu_B = e\hbar/2m_e = 5.8 \times 10^{-5} \text{ eV/T}$.

1. a) An electron is trapped inside a sphere of radius R by a potential that is zero for $r < R$ and infinite for $r > R$. Write down its ground state energy, and state the boundary conditions that lead to this. Calculate this energy for a sphere of radius $R = 3 \text{ nm}$, made of solid CdTe in which electrons have an effective mass $m^* = 0.096 m_e$.

[5 marks]

- b) Without any detailed algebra, explain the form of the expression

$$T \simeq \exp \left[-2 \int_a^b \sqrt{\frac{2m}{\hbar^2} (V(x) - E)} dx \right]$$

for the probability of a particle of energy E to tunnel through a smoothly varying barrier $V(x)$. Your answer should define the points a, b .

[5 marks]

- c) Write down the spin operators \hat{S}_x , \hat{S}_y , and \hat{S}_z in terms of the Pauli matrices and hence verify that

$$[\hat{S}_z, \hat{S}_y] = -i\hbar\hat{S}_x.$$

[5 marks]

- d) A W boson has an intrinsic spin $S = 1$ and is in a state with orbital angular momentum $L = 1$. What are its possible values of the quantum numbers J and M_J describing its total angular momentum?

[5 marks]

- e) The spins of two electrons are in the state

$$\phi(1, 2) = \frac{1}{2} (\alpha_x(1)\alpha_x(2) - \alpha_x(1)\beta_x(2) - \beta_x(1)\alpha_x(2) + \beta_x(1)\beta_x(2)).$$

Is this an "entangled" state? Justify your answer.

[5 marks]

2. a) Define the terms “quantum dot”, “quantum wire” and “quantum well”.
[5 marks]
- b) A quantum wire is constructed with a rectangular cross section, with sides a and $2a$ in the x and y directions. The wire is surrounded on all sides by an insulator which you can treat as providing an infinite potential energy for electrons. Consider a long piece of this wire, with length $L \gg a$ in the z direction. Write down the the general form of the energy eigenvalues of an electron in this wire. What are the energy levels of the ground state and the first two excitations of the xy motion?
[7 marks]
- c) Find the density of states for electrons in the wire, dn/dk_z , as a function of k_z , the wavenumber in the z direction. Convert this to the density of states as a function of energy, dn/dE , for a particular eigenvalue of the xy motion. Sketch the total density of states as a function of energy.
[13 marks]

3. a) Explain briefly the origin of the spin-orbit interaction,

$$\hat{H}_{so} = A \hat{\mathbf{L}} \cdot \hat{\mathbf{S}},$$

in a one-electron atom.

[5 marks]

- b) Show that eigenstates of the spin-orbit interaction \hat{H}_{so} do not in general have definite values for the quantum numbers m_l and m_s . Show also that these eigenstates can have definite values for m_j . You may assume the commutation rules for the components of any angular momentum operator, e.g.

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z.$$

Rewrite the spin-orbit interaction in a form that shows that eigenstates of \hat{H}_{so} can have a definite value for the quantum number j .

[10 marks]

- c) The outer electron of a neutral potassium atom is in a $3g$ orbital. Show that there are two possible values for its energy, and find the difference between these values in terms of the constant A .

[5 marks]

- d) Describe what happens to these two $3g$ levels when the atom is placed in a weak magnetic field B . Calculate the splitting between the states of the lower level for a field of strength $B = 1$ T.

[5 marks]

4. a) A quantum device is constructed so that it can copy the circular polarisation state of a photon by emitting a second photon with the same circular polarisation. If R and L denote the right- and left-handed polarisation states, the action of the copier can be represented by

$$\hat{X} R(1) = R(1) R(2), \quad \text{and} \quad \hat{X} L(1) = L(1) L(2).$$

Show that the device fails to produce an identical copy of a photon that is in a general polarisation state. Show also that the resulting pair of photons is in an entangled state.

[7 marks]

- b) Alice and Bob prepare pairs of photons in the entangled state

$$\Psi(1, 2) = \frac{1}{\sqrt{2}} \left(R(1)R(2) + L(1)L(2) \right),$$

which they use to encrypt messages by sending photon 1 to Alice and photon 2 to Bob. Each of them measures their photons with analysers that distinguish vertical and horizontal polarisations, with eigenstates

$$V = \frac{1}{\sqrt{2}} (R + L), \quad \text{and} \quad H = \frac{1}{\sqrt{2}} (R - L).$$

Show that Bob is able to deduce the results of Alice's measurements.

[5 marks]

- c) Eve breaks into their communications and uses the device in part (a) to copy the polarisation of Bob's photons, leaving them with three photons in the state,

$$\hat{X}\Psi(1, 2) = \frac{1}{\sqrt{2}} \left(R(1)R(2)R(3) + L(1)L(2)L(3) \right),$$

where photon 3 is Eve's copy. Suppose that all three people measure their photons with analysers that distinguish vertical and horizontal polarisations, as described above. Show that Eve gets no information about the state of polarisation of Bob's photons.

[8 marks]

- d) Explain why Alice and Bob will suspect that they have an eavesdropper.

[5 marks]

END OF EXAMINATION PAPER

PHYSICAL CONSTANTS AND CONVERSION FACTORS

SYMBOL	DESCRIPTION	NUMERICAL VALUE
c	Velocity of light in vacuum	$299\,792\,458\text{ m s}^{-1}$, exactly
μ_0	Permeability of vacuum	$4\pi \times 10^{-7}\text{ N A}^{-2}$, exactly
ϵ_0	Permittivity of vacuum where $c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$	$8.854 \times 10^{-12}\text{ C}^2\text{ N}^{-1}\text{ m}^{-2}$
h	Planck constant	$6.626 \times 10^{-34}\text{ J s}$
\hbar	$h/2\pi$	$1.055 \times 10^{-34}\text{ J s}$
G	Gravitational constant	$6.674 \times 10^{-11}\text{ m}^3\text{ kg}^{-1}\text{ s}^{-2}$
e	Elementary charge	$1.602 \times 10^{-19}\text{ C}$
eV	Electronvolt	$1.602 \times 10^{-19}\text{ J}$
α	Fine-structure constant, $\frac{e^2}{4\pi\epsilon_0\hbar c}$	$\frac{1}{137.0}$
m_e	Electron mass	$9.109 \times 10^{-31}\text{ kg}$
$m_e c^2$	Electron rest-mass energy	0.511 MeV
μ_B	Bohr magneton, $\frac{e\hbar}{2m_e}$	$9.274 \times 10^{-24}\text{ J T}^{-1}$
R_∞	Rydberg energy $\frac{\alpha^2 m_e c^2}{2}$	13.61 eV
a_0	Bohr radius $\frac{1}{\alpha} \frac{\hbar}{m_e c}$	$0.5292 \times 10^{-10}\text{ m}$
Å	Angstrom	10^{-10} m
m_p	Proton mass	$1.673 \times 10^{-27}\text{ kg}$
$m_p c^2$	Proton rest-mass energy	938.272 MeV
$m_n c^2$	Neutron rest-mass energy	939.565 MeV
μ_N	Nuclear magneton, $\frac{e\hbar}{2m_p}$	$5.051 \times 10^{-27}\text{ J T}^{-1}$
fm	Femtometre or fermi	10^{-15} m
b	Barn	10^{-28} m^2
u	Atomic mass unit, $\frac{1}{12} m(^{12}\text{C atom})$	$1.661 \times 10^{-27}\text{ kg}$
N_A	Avogadro constant, atoms in gram mol	$6.022 \times 10^{23}\text{ mol}^{-1}$
T_t	Triple-point temperature	273.16 K, exactly
k	Boltzmann constant	$1.381 \times 10^{-23}\text{ J K}^{-1}$
R	Molar gas constant, $N_A k$	$8.314\text{ J mol}^{-1}\text{ K}^{-1}$
σ	Stefan-Boltzmann constant, $\frac{\pi^2}{60} \frac{k^4}{\hbar^3 c^2}$	$5.670 \times 10^{-8}\text{ W m}^{-2}\text{ K}^{-4}$
M_E	Mass of Earth	$5.97 \times 10^{24}\text{ kg}$
R_E	Mean radius of Earth	$6.4 \times 10^6\text{ m}$
g	Standard acceleration of gravity	$9.806\,65\text{ m s}^{-2}$, exactly
atm	Standard atmosphere	101 325 Pa, exactly
M_\odot	Solar mass	$1.989 \times 10^{30}\text{ kg}$
R_\odot	Solar radius	$6.96 \times 10^8\text{ m}$
L_\odot	Solar luminosity	$3.84 \times 10^{26}\text{ W}$
T_\odot	Solar effective temperature	$5.8 \times 10^3\text{ K}$
AU	Astronomical unit, mean Earth-Sun distance	$1.496 \times 10^{11}\text{ m}$
pc	Parsec	$3.086 \times 10^{16}\text{ m}$
	Year	$3.156 \times 10^7\text{ s}$