ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Applications of Quantum Physics

19th January 2011, 9.45 a.m. - 11.15 a.m.

Answer **ALL** parts of question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

Here α_z and β_z denote the eigenstates of $\widehat{S}_z$ with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively. You may use the standard set of Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

You may also use the Landé g-factor,

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)},$$

and the fact that $ec = 0.3$ eV/nm/T.

1. a) Estimate the probability for a neutron to tunnel through a square barrier 2 fm wide when the energy of the neutron is 40 MeV below the top of the barrier.

[5 marks]

b) An electron is in the ground state of a one-dimensional harmonic oscillator well. It is subject to a weak electric field which leads to an additional term

$$\widehat{H}_1 = e\mathcal{E}\,x$$

in the electron's Hamiltonian. Show that to first order in $\mathcal{E}$, the electric field has no effect on the energy of the electron. [The ground-state wave function of the harmonic oscillator has the form $\psi_0(x) = N\exp(-x^2/(2b^2))$ where N and b are constants.]

[5 marks]

c) Show that

$$\alpha_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$$

is an eigenstate of the spin operator $\widehat{S}_y$ and find its eigenvalue.
Show also that α_y is not an eigenstate of $\widehat{S}_x$.

[5 marks]

d) The two electrons in an atom are excited into a state where one has angular momentum quantum number $j_1 = 3/2$ and the other has $j_2 = 9/2$. What are the possible values of J (the total angular momentum quantum number)?

[5 marks]

e) The spins of two electrons are in the state

$$\psi(1,2) = \frac{1}{\sqrt{2}}\Big(\alpha_z(1)\,\beta_z(2) - \beta_z(1)\,\alpha_z(2)\Big).$$

Why do we describe this as an "entangled" state?

[5 marks]

2. A quantum dot consists of a cuboid block of undoped silicon, with sides of length a, a and $a/\sqrt{2}$. It is surrounded by insulating walls, where the potential energy of the electron can be assumed to be infinite. The potential energy of an electron inside the dot is zero.

a) Find the first three energy levels of a single electron in the dot. What are the degeneracies of these levels? What "magic" numbers of electrons correspond to the first three closed shells?

[10 marks]

b) Find the ground-state energy of the electron for a block with $a = 10$ nm, taking the effective mass of an electron in silicon to be $m^* = 0.2\, m_e$. Where in the electromagnetic spectrum would you expect to see radiation from transitions between states in this dot?

[4 marks]

c) The dot is placed between two electrical contacts (the drain and source) and is separated from them by thin insulating layers. It is also surrounded by a gate, which can be used to change the constant potential inside the dot. The drain and source are held at the same voltage. Explain why the dot will conduct current only for certain discrete values of the gate voltage. Describe the pattern you expect for the spacings between these values.

[11 marks]

3. An electron with zero orbital angular momentum is placed in a magnetic field that points in the $+z$ direction. The magnetic moment of the electron is

$$\widehat{\mu} = -\frac{e}{m_e}\widehat{\mathbf{S}}.$$

At time $t = 0$, the electron is in the eigenstate of $\widehat{S}_y$,

$$\alpha_y = \frac{1}{\sqrt{2}}\left(\alpha_z + \mathrm{i}\,\beta_z\right).$$

a) Write down the time-dependent Schrödinger equation describing the interaction of this moment with the field. Hence show that, at time t, the spin state of the electron is

$$\chi(t) = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\omega t}\alpha_z + \mathrm{i}\,\mathrm{e}^{+\mathrm{i}\omega t}\beta_z\right),$$

and find an expression for ω.

[10 marks]

b) Show that the expectation value of $\widehat{S}_y$ in this state at time t is given by

$$\langle\widehat{S}_y\rangle = \frac{\hbar}{2}\cos(2\omega t).$$

[8 marks]

c) Find the probability for a measurement of $\widehat{S}_y$ at time t to give the value $+\hbar/2$.

[7 marks]

4. a) An atom is in an energy level with total orbital angular momentum quantum number L and total spin quantum number S. These are coupled to a total angular momentum quantum number J. The atom is placed in a weak magnetic field.

Write down an expression for the magnetic energies of the states corresponding to this level. Without giving a detailed algebraic derivation, explain the origin of the Landé g-factor in this expression. Why does the magnetic field need to be weak for this to be a good approximation?

[8 marks]

b) The low-energy states of Ti^{3+} ions have $L = 2$ and $S = 1/2$. The spin-orbit interaction Hamiltonian for the ions can be taken to have the form

$$\widehat{H}_{so} = A\,\widehat{\mathbf{L}} \cdot \widehat{\mathbf{S}},$$

where A is a constant. List the possible values of J and determine the resulting energy shifts for each of these states.

[7 marks]

c) Sketch the energy-level diagram of this system in a weak magnetic field. Indicate the quantum numbers of all levels. Find the splittings between the levels for a magnetic field of 2 T.

[6 marks]

d) Ti^{3+} ions are placed in a magnetic field that is so strong that the effect of the spin-orbit interaction can be neglected. Sketch the energy-level diagram for the states with $L = 2$ and $S = 1/2$ in this field.

[4 marks]

END OF EXAMINATION PAPER

PHYSICAL CONSTANTS AND CONVERSION FACTORS

SYMBOL	DESCRIPTION	NUMERICAL VALUE
c	Velocity of light in vacuum	$299\,792\,458$ m s^{-1}, exactly
μ_0	Permeability of vacuum	$4\pi \times 10^{-7}$ N A^{-2}, exactly
ϵ_0	Permittivity of vacuum where $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$	8.854×10^{-12} C^2 N^{-1} m^{-2}
h	Planck constant	6.626×10^{-34} J s
$\hbar$	$h/2\pi$	1.055×10^{-34} J s
G	Gravitational constant	6.674×10^{-11} m^3 kg^{-1} s^{-2}
e	Elementary charge	1.602×10^{-19} C
eV	Electronvolt	1.602×10^{-19} J
α	Fine-structure constant, $\frac{e^2}{4\pi\epsilon_0 \hbar c}$	$\frac{1}{137.0}$
m_e	Electron mass	9.109×10^{-31} kg
$m_e c^2$	Electron rest-mass energy	0.511 MeV
μ_B	Bohr magneton, $\frac{e\hbar}{2m_e}$	9.274×10^{-24} J T^{-1}
R_∞	Rydberg energy $\frac{\alpha^2 m_e c^2}{2}$	13.61 eV
a_0	Bohr radius $\frac{1}{\alpha}\frac{\hbar}{m_e c}$	0.5292×10^{-10} m
Å	Angstrom	10^{-10} m
m_p	Proton mass	1.673×10^{-27} kg
$m_p c^2$	Proton rest-mass energy	938.272 MeV
$m_n c^2$	Neutron rest-mass energy	939.565 MeV
μ_N	Nuclear magneton, $\frac{e\hbar}{2m_p}$	5.051×10^{-27} J T^{-1}
fm	Femtometre or fermi	10^{-15} m
b	Barn	10^{-28} m^2
u	Atomic mass unit, $\frac{1}{12} m(^{12}\text{C atom})$	1.661×10^{-27} kg
N_A	Avogadro constant, atoms in gram mol	6.022×10^{23} mol^{-1}
T_t	Triple-point temperature	273.16 K, exactly
k	Boltzmann constant	1.381×10^{-23} J K^{-1}
R	Molar gas constant, $N_A k$	8.314 J mol^{-1} K^{-1}
σ	Stefan-Boltzmann constant, $\frac{\pi^2}{60}\frac{k^4}{\hbar^3 c^2}$	5.670×10^{-8} W m^{-2} K^{-4}
M_E	Mass of Earth	5.97×10^{24} kg
R_E	Mean radius of Earth	6.4×10^6 m
g	Standard acceleration of gravity	$9.806\,65$ m s^{-2}, exactly
atm	Standard atmosphere	101 325 Pa, exactly
$M_\odot$	Solar mass	1.989×10^{30} kg
$R_\odot$	Solar radius	6.96×10^8 m
$L_\odot$	Solar luminosity	3.84×10^{26} W
$T_\odot$	Solar effective temperature	5.8×10^3 K
AU	Astronomical unit, mean Earth-Sun distance	1.496×10^{11} m
pc	Parsec	3.086×10^{16} m
	Year	3.156×10^7 s