PHYS30101

ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Applications of Quantum Physics

28th January 2010, 9.45-11.15

Answer **ALL** parts of question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

P.T.O.

You may use the standard set of Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

You may also use the Landé g-factor,

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)},$$

and the fact that $ec = 0.3$ eV/nm/T.

1. (a) Estimate the probability for an electron to tunnel through a square barrier 1 nm wide when the energy of the electron is 2 eV below the top of the barrier.
[5 marks]

(b) An electron is trapped inside an elongated quantum dot by the potential

$$V(x, y, z) = \frac{1}{2} k \left(x^2 + y^2 + 4z^2 \right).$$

Write down its first three energy levels and state their degeneracies. [5 marks]

(c) Write down the spin operators $\widehat{S}_x$, $\widehat{S}_y$, and $\widehat{S}_z$ in terms of the Pauli matrices and hence verify that

$$\left[\widehat{S}_z, \widehat{S}_x\right] = \mathrm{i}\hbar \widehat{S}_y.$$

[5 marks]

(d) The quarks in an excited baryon have a total orbital and spin angular-momentum quantum numbers $L = 1$ and $S = 3/2$, respectively. What are the allowed values of J (the quantum number that gives the eigenvalue the total angular momentum)?
[5 marks]

(e) Find the interaction energy (in eV) of the spin of an electron with the magnetic field of the world's strongest magnet, $B = 45$ T. [5 marks]

P.T.O.

2. (a) Without any detailed algebra, explain the origin of the formula

$$T \simeq \exp\left[-2\int_a^b \sqrt{\frac{2m}{\hbar^2}\Big(V(x)-E\Big)}\,\mathrm{d}x\right]$$

for the probability of a particle of energy E to tunnel through a barrier $V(x)$. Your answer should state briefly the conditions under which this is a good approximation, and define the points a, b. [8 marks]

(b) In nuclear α decay, a ^{4}He nucleus (atomic number $Z_1 = 2$) tunnels out through the repulsive Coulomb potential between it and the final nucleus (atomic number Z_2). Assuming that the attractive nuclear force forms a square well of radius R_N, find an expression for the tunnelling probability for an alpha particle of energy E to escape from the nucleus. [9 marks]

[You may use the standard integral,

$$\int \left(\frac{1}{x}-1\right)^{1/2} \mathrm{d}x = \sqrt{x(1-x)} - \cos^{-1}\sqrt{x} + C.\ ]$$

(c) Bismuth isotopes can undergo α decay to thallium ($Z_2 = 81$). One isotope, bismuth-211, emits α particles with energy 6.8 MeV and has a lifetime of 150 s. Another, bismuth-209, emits α particles with energy 3.2 MeV, and has a lifetime of 6.0×10^{26} s. You may assume that the radii of the nuclei involved are small compared to any other relevant distances in the problem, and you should ignore the recoil of the heavy nuclei. Use your result above to explain this factor $\sim 10^{24}$ difference in lifetimes. [8 marks]

P.T.O.

3. (a) Explain the origin of the spin-orbit coupling in a one-electron atom, commenting on the sign of the interaction. [6 marks]

(b) The spin-orbit interaction energy operator has the form

$$\widehat{H}_{so} = f(r)\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}.$$

Rewrite this in a form that shows that eigenstates of the total angular momentum operator $\widehat{\mathbf{J}}^2$ are not mixed by this interaction. [3 marks]

(c) A potassium atom has one electron outside a closed shell. Its lowest $l = 1$ level is split into two levels with an energy difference of 7.2×10^{-3} eV. State the values of the quantum number j for these levels, indicating which has the lower energy. Determine the value (in eV) of the radial matrix element

$$\mathcal{E}_{so} = \langle f(r) \rangle \hbar^2.$$

[8 marks]

(d) Sketch the energy-level diagram of this $l = 1$ system in a weak magnetic field. Indicate the quantum numbers of all levels. Find the splittings between the levels for a magnetic field of 0.5 T. [8 marks]

4. (a) The normalised eigenvectors of the operator $\widehat{S}_y$ for a spin-1/2 particle are

$$\alpha_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \quad \text{and} \quad \beta_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}.$$

Verify that α_y is indeed an eigenvector and that it is normalised. [4 marks]

(b) Two electrons are in a state $\psi(1,2)$ with a total spin of zero. Write down an expression for this state in terms of $\alpha_z(i)$ and $\beta_z(i)$, the eigenvectors of $\widehat{S}_z$ for the two electrons ($i = 1, 2$). Express the state $\psi(1,2)$ in terms of the vectors $\alpha_y(i)$ and $\beta_y(i)$ given above. Explain briefly why we say that these two electrons are in an "entangled" state. [13 marks]

(c) Explain how Alice and Bob can use pairs of electrons prepared in the state $\psi(1,2)$ to set up a secure key for the exchange of encrypted messages. How could they detect whether a third party had been eavesdropping? [8 marks]

END OF EXAMINATION PAPER

PHYSICAL CONSTANTS AND CONVERSION FACTORS

SYMBOL	DESCRIPTION	NUMERICAL VALUE
c	Velocity of light in vacuum	$299\,792\,458$ m s^{-1}, exactly
μ_0	Permeability of vacuum	$4\pi \times 10^{-7}$ N A^{-2}, exactly
ϵ_0	Permittivity of vacuum where $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$	8.854×10^{-12} C^2 N^{-1} m^{-2}
h	Planck constant	6.626×10^{-34} J s
$\hbar$	$h/2\pi$	1.055×10^{-34} J s
G	Gravitational constant	6.674×10^{-11} m^3 kg^{-1} s^{-2}
e	Elementary charge	1.602×10^{-19} C
eV	Electronvolt	1.602×10^{-19} J
α	Fine-structure constant, $\frac{e^2}{4\pi\epsilon_0\hbar c}$	$\frac{1}{137.0}$
m_e	Electron mass	9.109×10^{-31} kg
$m_e c^2$	Electron rest-mass energy	0.511 MeV
μ_B	Bohr magneton, $\frac{e\hbar}{2m_e}$	9.274×10^{-24} J T^{-1}
R_∞	Rydberg energy $\frac{\alpha^2 m_e c^2}{2}$	13.61 eV
a_0	Bohr radius $\frac{1}{\alpha}\frac{\hbar}{m_e c}$	0.5292×10^{-10} m
Å	Angstrom	10^{-10} m
m_p	Proton mass	1.673×10^{-27} kg
$m_p c^2$	Proton rest-mass energy	938.272 MeV
$m_n c^2$	Neutron rest-mass energy	939.565 MeV
μ_N	Nuclear magneton, $\frac{e\hbar}{2m_p}$	5.051×10^{-27} J T^{-1}
fm	Femtometre or fermi	10^{-15} m
b	Barn	10^{-28} m^2
u	Atomic mass unit, $\frac{1}{12}m(^{12}\text{C atom})$	1.661×10^{-27} kg
N_A	Avogadro constant, atoms in gram mol	6.022×10^{23} mol^{-1}
T_t	Triple-point temperature	273.16 K, exactly
k	Boltzmann constant	1.381×10^{-23} J K^{-1}
R	Molar gas constant, $N_A k$	8.314 J mol^{-1} K^{-1}
σ	Stefan-Boltzmann constant, $\frac{\pi^2}{60}\frac{k^4}{\hbar^3 c^2}$	5.670×10^{-8} W m^{-2} K^{-4}
M_E	Mass of Earth	5.97×10^{24} kg
R_E	Mean radius of Earth	6.4×10^6 m
g	Standard acceleration of gravity	$9.806\,65$ m s^{-2}, exactly
atm	Standard atmosphere	101 325 Pa, exactly
$M_\odot$	Solar mass	1.989×10^{30} kg
$R_\odot$	Solar radius	6.96×10^8 m
$L_\odot$	Solar luminosity	3.84×10^{26} W
$T_\odot$	Solar effective temperature	5.8×10^3 K
AU	Astronomical unit, mean Earth-Sun distance	1.496×10^{11} m
pc	Parsec	3.086×10^{16} m
	Year	3.156×10^7 s