Generalized Calderón–Ryaben'kii's Potentials

Sergei V. Utyuzhnikov
School of Mechanical Aerospace and Civil Engineering,
University of Manchester, P.O. Box 88, Manchester, M60 1QD, UK.

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Abstract

Calderón–Ryaben'kii potentials provide the foundation for the difference potential method, which is an efficient way for solving boundary value problems in arbitrary domains. This method allows us to reduce a uniquely solvable and well–posed boundary value problem to a pseudo–differential boundary equation. The general theory of Calderón–Ryaben'kii potentials is considered via the theory of distributions. The definition of Calderón–Ryaben'kii potentials is based on the notion of a clear trace. The criterion of the clear trace is formulated. Partial differential equations of the first order and the second order are considered as particular examples. On the basis of the Calderón–Ryaben'kii potential theory, a solution of the active sound control problem is obtained in a general formulation. For the first time, the solution of the problem takes into account the feedback of the active shielding sources on the input (measurement) data. The exact transfer of the boundary conditions from the original boundary to an artificial boundary is also considered.

1 Introduction

Calderón–Ryaben'kii's potentials provide the foundation of the Difference Potential Method (DPM) [1]. This method allows us to reduce a uniquely solvable and well–posed boundary value problem (BVP) in a quite arbitrary domain to a pseudo–differential boundary equation. The replacement of a BVP by a boundary equation is very attractive; the boundary equation is very beneficial for numerically solving the BVP because it drastically diminishes the number of unknown (grid) variables. The classical example of such a reduction is the Fredholm integral equation for the Laplace and Helmholtz equations. In complex analysis this reduction is given by a Cauchy–type integral. It is worth noting that on the basis of Green's formula the very efficient boundary element method (BEM) was developed, see e.g. [2]). Nevertheless, the BEMs have a relatively limited area of application.

Calderón was the first to reduce a BVP for a general linear differential elliptic equation to a pseudo–differential boundary equation [3]. This work was further developed by Seeley [4] who, in particular, showed that the Calderón projection of an elliptic operator is represented by a pseudo–differential equation. Later Hörmander [5] demonstrated that the Calderón theory, in fact, is not limited by elliptic problems. Some drawbacks of these formulations were related to their complexity and the absence of a robust method for their solvability. It was the DPM by Ryaben'kii [1] that provided an approach for the formulation of the boundary equation in a general finite–difference form. In [1], Calderón's potentials are modified to be approximated via finite–difference potentials based on the solution of an auxiliary classical boundary value problem. Ryaben'kii introduces an auxiliary "simple" domain containing the original domain. Although the auxiliary domain it is not necessary for the reduction of a BVP to a boundary equation, it is very important from the standpoint of applications. Ryaben'kii effectively reduces the solution of the boundary equation to the solution of a BVP in the auxiliary domain which is much simpler than that in the original domain. Apart from its original intended role, the auxiliary domain appears to play a significant role in different applications such as active noise shielding (AS) and artificial boundary conditions (ABC) both of which are considered in this paper.

The DPM became a powerful mathematical tool for solving complicated problems of mathematical physics; some examples are given in the monograph [1]. In the papers by Ryaben'kii and his co-authors the most attention is devoted to the development of the difference potentials, the numerical methods for solving the boundary equation and applications. Apart from the finite–difference formulation, in [1] the DPM is also considered in the differential classical form. General aspects of the theory in continuous and discrete spaces are addressed in [6]. Some extension of the DPM formalism to the linear Helmholtz–type equations with discontinuous solutions is given in [8], where this theory is applied to the active sound control problem. In [1], it is proven by Kamenetskii that the potentials introduced by Seeley [4] for elliptic equations are equivalent to the Ryaben'kii potentials. In turn, Kamenetskii [7] proved that the Calderón potentials, cited in [4], can be reduced step by step on the discrete level to one form of the difference potentials suggested by Reznik [1]. It is to be noted that the analogue between the Calderón and Ryaben'kii potentials is not obvious and, historically, it was not immediately observed. Therefore, for a long time the theory of the DPM was developed by Ryaben'kii and his co-workers completely independently from the theory of the Calderón potentials.

In this paper, the theory of the Calderón–Ryaben'kii potentials is extended to a generalized formulation based on the theory of distributions (see e.g. [9], [10]). Under the Calderón–Ryaben'kii potentials we understand the extension of the Ryaben'kii difference potentials to continuous spaces. It is to be noted that the name adopted for the potentials is not traditional in the literature. Meanwhile, we believe that such a name is the most proper. It appears that the difference potential theory can naturally be formulated via the formalism of distributions. The weak formulation of the term Calderón–Ryaben'kii potentials is suggested in the current paper. It allows us to exploit all the advantages of the generalized formulations, including extending the theory to piecewise continuous functions and generalized non-regular functions. In particular, it is important from the standpoint of the application of the theory to physical problems. The key proposition about validity of the generalized potential definition is given in this paper. The Calderón–Ryaben'kii potentials are based on the notion of a clear trace. Here, the criterion of the clear trace is formulated. The generalized formulation of the Calderón–Ryaben'kii potentials can be useful for understanding the algorithms and potential applications of the DPM.

The active noise shielding problem (see e.g. [11], [12]) is considered as an application example in this paper. This problem is addressed using the DPM in [1], [13], [8], [14], [15], [16], [17], [18] and by using the theory of distributions in [15], [19]. The solution of this problem for a linear analogue of the Helmholtz equation with variable coefficients is obtained in [8], [15]. For an arbitrary linear problem of first order equations the AS is obtained in [19] in the form of a simple–layer potential. In the current paper, the general solution of the AS problem is first obtained for a general differential operator via the theory of the generalized potentials. In particular, the solution gives the AS secondary terms for the Helmholtz equation and the Euler acoustics equations, which coincide with the results obtained in [8] and [19], respectively. Finally, the feedback of the secondary sources is taken into account. It is shown that in this case the solution of the AS problem might require the solution of some additional BVP.

Another application example is related to the boundary conditions to be set on an artificial boundary. In many applications, it is desirable to restrict the solution of the problem in the original domain to the sub-domain where the right–hand side is supported. It makes sense, for instance, if the new domain can be chosen much smaller than the original one. In this case, were are required to set the boundary conditions on the artificial boundary. Such boundary conditions are called artificial boundary conditions [20]. The DPM can be used to provide the exact transfer of the boundary conditions from the remote boundary to the artificial boundary [1], [20].

The paper is organized in the following way. The Calderón–Ryaben'kii potentials are first introduced for a generally formulated linear BVP, which is supposed to be both uniquely solvable and well–posed. The solution of the BVP is considered in the generalized (weak) sense. The definition of the potentials is then formulated via the theory of distributions, which is strongly based on the notion of the clear trace introduced by Ryaben'kii. Then, the main properties of the Calderón–Ryaben'kii potentials are considered. They include the generalized Green's identity, decomposition of the trace, reduction of the original BVP to an equivalent boundary pseudo–differential equation. The criterion of the clear trace is then given. It is shown that the potential can be obtained via the solution of some BVP with respect to some density of the potential on the right–hand side. First order and second order differential equations are considered as particular cases. The application of the Calderón–Ryaben'kii potentials is demonstrated on the examples of the AS problem and artificial boundary conditions.

2 The generalized formulation of the Calderón–Ryaben'kii potentials

2.1 Statement of the problem

First, let us introduce some domain D^0: $\overline{D}^0 \subseteq \mathbb{R}^m$ with smooth boundary Γ^0 and a sub-domain $D : \overline{D} \subset D^0$, having smooth boundary Γ.

Let us now consider the following linear BVP:

$$LU = f, \tag{1}$$

$$U \in \Xi_{D^0}, \tag{2}$$

where L is some differential operator of order k with sufficiently smooth coefficients, $U \in \mathbb{R}^p$, $f \in \mathbb{R}^p$. Let a linear functional space Ξ_{D^0} be such that the solution of the homogeneous BVP (1), (2) with $f = 0$ is unique and trivial: $U \equiv 0$. To avoid any possible confusion, it is supposed that the boundary conditions are locally formulated at the boundary Γ^0. We say that a function U is a generalized solution of BVP (1), (2) if $\langle LU, \Phi\rangle = \langle f, \Phi\rangle$ for any test function $\Phi(\overline{D}^0) \in C_0^\infty(\overline{D}^0)$. Here, $\langle f, \Phi\rangle$ denotes a linear continuous functional associated with a given generalized function f.

Suppose that in (1) the right–hand side $f \in F_{D^0}$ where the space F_{D^0} is defined such that the solution of BVP (1), (2) exists. It is easy to see that if the solution of BVP (1), (2) exists, then it is unique. Thus, the spaces Ξ_{D^0} and F_{D^0} are isomorphic each other. In addition, we require that if $f \in F_{D^0}$, then $\theta(D)f \in F_{D^0}$, where $\theta(D)$ is the Heaviside–type characteristic function equal to 1 in D and 0 outside.

Along with a generalized function ϕ we introduce a local element [9] ϕ_Ω of $\phi \in \Xi_{D^0}$ on Ω ($\Omega \subset D^0$) as the restriction of ϕ to Ω. We also consider the following additional linear spaces:

$$F_\Omega = \{f_\Omega |\ f \in F_{D^0}\}, \tag{3}$$

$$\Xi_\Omega = \{U_\Omega |\ U \in \Xi_{D^0}\}. \tag{4}$$

We assume further that the space Ξ_{D^0} is the space of piecewise bounded functions having generalized regular derivatives up to order k both on D and $D^- \stackrel{def}{=} D^0 \backslash \overline{D}$. In addition, we require that any function from Ξ_{D^0} is bounded along with its $k-1$ derivatives in the appropriate norms.

The specifications on to BVP (1), (2), described above, are sufficient for further analysis. However, we introduce some additional conditions in order make this analysis more concrete. Let us suppose that

$$\Xi_{D^0} \subset H^s(D) \cap H_0^s(D^-),$$

where $s > k - 1/2$, H^s and H_0^s are Sobolev spaces. Thus, $f \in H_{loc}^{s-k}(D) \cap H_{loc}^{s-k}(D^-)$.

We assume that BVP (1), (2) is well–posed according to Hadamard; i.e. we require the following estimate:

$$\parallel U_D \parallel^2_{H^s} + \parallel U_{D^-} \parallel^2_{H^s} < C(\parallel LU_{|D} \parallel^2_{H^{s-k}} + \parallel LU_{|D^-} \parallel^2_{H^{s-k}}),$$

where C is some positive constant. In addition, we suppose the space Ξ_{D^0} should not be degenerate. Thus, we assume that the boundary conditions are not over–determined. Thereby, they can contain a linear differential operator of order not greater than $k-1$, and they are not necessarily to be formulated on the entire boundary. In particular, the linear differential operator L in (1) can correspond to operators of first order or second order. For the sake of simplicity we will consider either a system of first–order equations or one higher–order equation.

The first order operator L is represented by

$$L := L_f \overset{def}{=} \sum_1^m A^i \frac{\partial}{\partial y^i} + B, \tag{5}$$

where $\{y^i\}$ $(i = 1, ..., m)$ is some Cartesian coordinate system; in (1) U and f are vector–functions with the dimension of p; A^i, B are $p \times p$ matrices: $A^i = A^i(\mathbf{y}) \in C^1(\overline{D}^0), B = B(\mathbf{y}) \in C(\overline{D}^0)$.

The following elliptic operator is the typical case of the second order operator:

$$L := L_s \overset{def}{=} \nabla(p\nabla) + q, \tag{6}$$

where $p \in C^1(\overline{D^0})$, $q \in C(\overline{D^0})$ and $p > 0$.

Since the solution of BVP (1), (2) is unique, there exists a Green's operator G inverse to the operator $L:\ F_{D^0} \rightarrow U_{D^0}$. Along with the operator G we can introduce the local Green's operator $G_D:\ F_D \rightarrow \Xi_D$ as follows:

$$U_D = G_D f_D \overset{def}{=} (G\theta(D)f)_{|D}\,,$$

We also introduce on D a differential operator $L_D V_D \overset{def}{=} LV_{|D}$ where $LV_{|D}$ is the restriction of the function LV to D.

2.2 Definition of Calderón–Ryaben'kii's potentials. Clear trace

We define an operator $P_{DD} : \Xi_D \rightarrow \Xi_D$ as follows.

Definition 2.1 For any $V \in \Xi_D$

$$P_{DD} V_D \overset{def}{=} V_D - G_D L_D V_D.$$

The function $P_{DD}V_D$ can also be rewritten as

$$P_{DD}V_D = (GLV)_{|D} - G_D L_D V_D = \\ (GLV - G\theta(D)LV)_{|D} = \Big(G\theta(\overline{D^-})LV\Big)_{|D}. \tag{7}$$

The function $P_{DD}V_D$ has the following important properties:

PROPOSITION 2.2 $\mathrm{ImP_{DD}} = \ker \mathrm{L_D}$ and $P_{DD}^2 = P_{DD}$.

Proof. From Definition 2.1 it immediately follows that $L_D P_{DD} W_D = 0_D$. In turn, if $L_D W_D = 0_D$, then $W_D \in \Xi_D$ and $P_{DD} W_D = W_D$. Hence, the operator P_{DD} is a projection: $P_{DD}^2 = P_{DD}$. □

Next, we introduce a trace operation as follows. Let Γ_ϵ^+ be smooth manifolds parallel to Γ in the sense of [9], [10, Ch. 2]: $\Gamma_\epsilon^+ \subset D$, $\Gamma_\epsilon^+ \to \Gamma$ if $\epsilon \to 0$. The trace operator $\mathrm{Tr}_\Gamma^+ : H^s(D) \to H^{s-1/2}(\Gamma)$ is given by

$$\mathrm{Tr}_\Gamma^+ U_D \stackrel{def}{=} \lim_{\epsilon \to 0} \mathrm{Tr}_{\Gamma_\epsilon^+} U_D,$$

where

$$\mathrm{Tr}_{\Gamma_\epsilon^+} U_D \stackrel{def}{=} U_D(\mathbf{x}), \quad \mathbf{x} \in \Gamma_\epsilon^+.$$

Similarly, in D^- we introduce the trace operator $\mathrm{Tr}_\Gamma^- : H^s(D^-) \to H^{s-1/2}(\Gamma)$ and

$$\mathrm{Tr}_\Gamma^- U_{D^-} \stackrel{def}{=} \lim_{\epsilon \to 0} U_{D^-}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_\epsilon^-.$$

If $\mathrm{Tr}_\Gamma^+ U_D = \mathrm{Tr}_\Gamma^- U_{D^-}$, then the trace on Γ is determined as

$$\mathrm{Tr}_\Gamma U_D \stackrel{def}{=} U(\Gamma) = \mathrm{Tr}_\Gamma^+ U_D = \mathrm{Tr}_\Gamma^- U_{D^-}. \tag{8}$$

If $\mathrm{Tr}_\Gamma U_D$ does not exist, then the function $U_\Gamma = U_{|\Gamma}$ has two values: $\mathrm{Tr}_\Gamma^+ U_D$ and $\mathrm{Tr}_\Gamma^- U_{D^-}$. Then, we introduce the following definition of the trace generalizing the definition for continuous functions:

$$\mathrm{Tr}_\Gamma U \stackrel{def}{=} \frac{1}{2}(\mathrm{Tr}_\Gamma^+ U_D + \mathrm{Tr}_\Gamma^- U_{D^-}). \tag{9}$$

We now give the definition of the *clear trace* first introduced by Ryaben'kii [1]. We consider some domain $\Omega \subset \Omega^0 \subseteq \mathbb{R}^m$ with a boundary $\gamma := \partial\Omega$. Let X_Ω, $\pi(\gamma)$ be Banach spaces of functions defined on Ω and γ, respectively. Then, let us consider some linear operator: $\mathrm{M} : \mathrm{X}_\Omega \to \mathrm{X}_\Omega$.

Definition 2.3 An operator $\mathrm{Tr}(\gamma) : X_\Omega \to \pi(\gamma)$ is called a clear trace operator, associated with the operator M, if for any two functions V and V' from X_Ω: $\mathrm{Tr}(\gamma)V = \mathrm{Tr}(\gamma)V' \in \pi(\gamma)$ it follows that $\mathrm{MV} = \mathrm{MV'}$. The pair $(\mathrm{Tr}(\gamma)V, \pi(\gamma))$ then creates *a clear trace* of M. The space $\pi(\gamma)$ is called the space of clear traces.

From the definition it follows that if $\mathrm{Tr}(\gamma)V = 0_\gamma$, then $MV = 0$. Thus, $\ker \mathrm{Tr}(\gamma) \subseteq \ker M$. Hence, the choice of the clear trace space is not unique because any subspace of some space $\pi(\Gamma)$ is also a space of clear traces. However, not every linear operator has a clear trace.

With this regard, following [1], we can introduce the notion of the minimal clear trace.

Definition 2.4 Let $\ker \mathrm{Tr}(\gamma) = \ker \mathrm{M}$. Then, the clear trace is called the minimal clear trace.

For practical applications it is better to chose the dimension of a clear trace space to be as small as possible. This issue will be discussed later.

Having applied Definition 2.3 to the operator P_{DD}, we are able to consider a clear trace operator $\mathrm{Tr}^+(\Gamma)$ associated with the operator P_{DD}: $\ker \mathrm{Tr}^+(\Gamma) \subseteq \ker P_{DD}$. Also, we have $\mathrm{Im\ P_{DD}} = \ker \mathrm{L_D}$, thus we have $\ker \mathrm{Tr}^+(\Gamma) \cap \ker \mathrm{L_D} = \{\emptyset\}$. It is worth noting that this property can be used as an alternative definition of the clear trace operator [1].

The operator $\mathrm{Tr}^+(\Gamma)$ in Definition 2.3 of the clear trace does not necessarily coincide with the Cauchy–data trace operator of the operator L:

$$\mathrm{Tr}_c^+(\Gamma) : \Xi_{D^0} \to \pi_c(\Gamma) \subset \oplus_0^{k-1} H^{s-1/2-j}(\Gamma),$$
$$\mathrm{Tr}_c^+(\Gamma)U = \mathrm{Tr}_\Gamma^+ \left(U, \frac{\partial U}{\partial \mathbf{n}}, ..., \frac{\partial^{k-1} U}{\partial \mathbf{n}^{k-1}} \right)^T,$$

where k is the order of the operator L, $\mathbf{n}$ is the external normal vector to the boundary Γ. The term a normal derivative refers to the regular normal derivative [9].

From the definition, it follows that the space $\pi_c(\Gamma)$ is the factor space of $\oplus_0^{k-1} H^{s-1/2-j}(\Gamma)$ with respect to $\ker P_\Gamma$. It is clear that, in contrast to the clear trace, the operator $\mathrm{Tr}_c^+(\Gamma)$ is not assigned to any other operator. Examples of clear traces for classical solutions can be found in [1]. In particular, the operator of the clear trace can be nonlocal.

Now, we are able to introduce the Calderón–Ryaben'kii potentials as follows.

Definition 2.5 Let $V \in \Xi_{D^0}$ and $\xi_\Gamma = \mathrm{Tr}^+(\Gamma)V_D \in \pi(\Gamma)$ where $\pi(\Gamma)$ is a space of clear traces of the operator P_{DD}. Then, a function

$$U_D = P_{D\Gamma}\xi_\Gamma \overset{def}{=} P_{DD}V_D \tag{10}$$

is called the potential with the density of ξ_Γ.

From the definition it follows that the potential $P_{D\Gamma}\xi_\Gamma$ does not depend on the complementation of V_D to $V_{D^0} \in \Xi_{D^0}$. Meanwhile, as mentioned above the complementary domain D^0 is important for practical applications [1].

Let us now introduce an operator $L_{\Gamma_+} \overset{def}{=} L\theta(D) - \theta(D)L : \Xi_{D^0} \to F_{D^0}$. The differential operator L_{Γ_+} acts in a neighborhood of Γ.

The operator $L_{\Gamma_+}U$ can be represented as:

$$L_{\Gamma_+}U = -\zeta_\Gamma A_\Gamma \,\mathrm{Tr}_c^+(\Gamma)U,$$

where A_Γ is a matrix with the dimension of $(k \times p) \times (k \times p)$, $\zeta_\Gamma \in \mathbb{R}^k$ is the following generalized vector–function:

$$\zeta_\Gamma \stackrel{def}{=} \left(\frac{\partial^{k-1}\delta(\Gamma)}{\partial \mathbf{n}^{k-1}}, ..., \frac{\partial^l \delta(\Gamma)}{\partial \mathbf{n}^l}, ..., \delta(\Gamma) \right). \tag{11}$$

Here, $\delta(\Gamma)$ is the surface delta–function.

If $L := L_f$, then $A_\Gamma = A_n \stackrel{def}{=} \sum_1^m A_i n^i$ where n_i are the coordinates of the vector $\mathbf{n}$. In turn, if $L := L_s$, then $A_\Gamma = p_\Gamma E$, where $p_\Gamma = p(\Gamma)$, E is the unit 2×2 matrix.

From the definition of the space F_{D^0} it follows that $L_{\Gamma_+} U \in F_{D^0}$ if $U \in \Xi_{D^0}$. Then, $GL_{\Gamma_+} U = \theta(D)U - GL_D U_D$. In the general case, the matrix A_Γ includes tangential differential operators. It should be noted that the matrix A_Γ might be singular; for example, it might have both the first row and the last column with only zero elements.

Now we prove the following important proposition.

PROPOSITION 2.6 The pair of $(\mathrm{Tr}_c^+(\Gamma), \pi_c(\Gamma))$ is a clear trace of the operator $P_{D\Gamma}$.

Proof. From the definition and relation (7), it follows that

$$P_{DD} V_D = (G\Psi)_{|D},$$

where $\Psi = LV - \theta(D)LV = L(\theta(\overline{D^-})V) + L_{\Gamma_+} \mathrm{Tr}_c^+(\Gamma)V$. It is easy to see that $\Psi \in F_{D^0}$ since $V_D \in \Xi_D$. Hence, Green's operator is determined. If $\mathrm{Tr}_c^+(\Gamma)V = 0_\Gamma$, then $P_{DD} = \theta(D^-)V_{|D} = 0_D$. Thus, $\ker \mathrm{Tr}_c^+(\Gamma) \subset \ker P_{DD}$. □

Proposition 2.6 implies validity of the definition of the potential $P_{D\Gamma}$. It is to be noted that if we consider the Cauchy data to the order of $k+1$:

$$\widehat{\mathrm{Tr}}_c^+(\Gamma)U = \mathrm{Tr}_\Gamma^+ \left(U, \frac{\partial U}{\partial \mathbf{n}}, ..., \frac{\partial^k U}{\partial \mathbf{n}^k} \right)^T \in \oplus_0^n H^{s-1/2-j}(\Gamma),$$

then they also provide a clear trace. However, it makes the application of the clear trace more complicated in this case which is not justified from a practical point of view.

A clear trace $\mathrm{Tr}^+(\Gamma)$ is called a clear trace of *canonical type* [1] if it is obtained by linear differential operators applied to $V_D \in \Xi_{D^0}$ at the boundary Γ. It is clear that $\mathrm{Tr}_c^+(\Gamma)$ represents an example of a clear trace of canonical type. Further we will only consider such type of clear traces. Let us now consider the main properties of the potentials introduced.

3 Properties of Calderón–Ryaben'kii's potentials

3.1 Generalized Green's identity and trace decomposition

We can rewrite definition (10) in the following form:

$$V_D = P_{D\Gamma} \xi_{\Gamma_+} + G_D L_D V_D, \tag{12}$$

where $\xi_{\Gamma_+} = \mathrm{Tr}^+(\Gamma)V_D$. Following [1], this equality is called the *generalized Green's identity.* If we set $L_D V_D = f_D$, then we obtain the generalized Green's formula

$$V_D = P_{D\Gamma}\xi_{\Gamma_+} + G_D f_D. \tag{13}$$

When applied to the Poisson equation, equality (13) gives us the well–known Green formula [1].

Along with the operator $P_{D\Gamma}$ we introduce a boundary operator $P_\Gamma : \pi(\Gamma) \to \pi(\Gamma)$ as follows.

Definition 3.1 Let $\xi_\Gamma \in \Xi_{D^0}$ and $\xi_\Gamma = \mathrm{Tr}^+(\Gamma)V$. Then

$$P_\Gamma \xi_\Gamma \stackrel{def}{=} \mathrm{Tr}^+(\Gamma)P_{D\Gamma}\xi_\Gamma. \tag{14}$$

From the definition and properties of P_{DD} it immediately follows that the operator P_Γ is a projection: $P_\Gamma^2 = P_\Gamma$.

The next two propositions give a decomposition of $\xi_{\Gamma_+} = \mathrm{Tr}^+(\Gamma)V$ if $V \in \Xi_{D^0}$.

Proposition 3.2 Let us consider $\xi_{\Gamma_+} = \xi^+_{\Gamma_+} \stackrel{def}{=} \mathrm{Tr}^+(\Gamma)U^+_D$ where U^+_D is defined such that $LU^+ = f^+$, $U^+ \in \Xi_{D^0}$ and $\mathrm{supp}\, f^+ \subset D$. Then, $U^+_D = G_D f^+_D$, $P_{D\Gamma}\xi^+_{\Gamma_+} = 0_D$ and $P_\Gamma \xi^+_{\Gamma_+} = 0_\Gamma$.

Proof. It immediately follows from the chain:

$$G_D f^+_D = G_D L U^+_{|D} = \left(GLU^+\right)_{|D} = U^+_D$$

The last statement of the proposition follows from Green's formula (13). □

Thus, $G_D L_D U_D = U_D$. Hence, $L_D G_D L_D U_D = L_D U_D$ and the operator L_D is semi-inverse to the operator G_D [1].

Proposition 3.3 Let us now consider $\xi_{\Gamma_+} = \xi^-_{\Gamma_+} \stackrel{def}{=} \mathrm{Tr}^+(\Gamma)U^-_D$ where U^-_D such that $LU^- = \overline{f}^{\,-}$, $U^- \in \Xi_{D^0}$ and $\mathrm{supp}\,\overline{f}^{\,-} \subset \overline{D^-}$. Then, $U^-_D = P_{DD}U^-_D$ and $P_\Gamma \xi^-_{\Gamma_+} = \xi^-_{\Gamma_+}$.

Proof. It is easy to see that

$$P_{DD}U^-_D = P_{D\Gamma}\xi^-_{\Gamma_+} = (Gf^-)_{|D} = U^-_D \tag{15}$$

Then, having taken $\mathrm{Tr}^+(\Gamma)$ from both sides of (15), we obtain

$$P_\Gamma \xi^-_{\Gamma_+} = \xi^-_{\Gamma_+}.$$

□

Corollary 1. If in BVP (1), (2) $f = f^+ + \overline{f}^{\,-}$ ($\mathrm{supp}\, f^+ \in D$, $\mathrm{supp}\,\overline{f}^{\,-} \in \overline{D}^{\,-}$), $U \in \Xi_{D^0}$, $\xi_\Gamma = \mathrm{Tr}^+(\Gamma)U_D$, then $P_\Gamma \xi_{\Gamma_+} = \xi^-_{\Gamma_+}$. This immediately follows from the linearity of the problem.

Corollary 2. The space $\pi(\Gamma)$ of clear trace of Ξ_{D^0} onto the boundary Γ is decomposed into a direct sum of two subspaces: $\pi(\Gamma) = \ker P_\Gamma \oplus \mathrm{Im}\ \mathrm{P}_\Gamma$.

Corollary 3. If $(\xi_\Gamma, \pi(\Gamma))$ is the minimal clear trace, then $P_\Gamma \xi_\Gamma = \xi_\Gamma$ [1]. It immediately follows from the definition of the minimal clear trace and Corollary 2.

We note here that, generally speaking, $\xi^-_{\Gamma_+}$ is not fully determined by U_{D^-} because the function $U \in \Xi_{D^0}$ can be discontinuous on Γ and $\mathrm{supp}\,\overline{f}^{\,-} \subset D^- \cup \Gamma$.

3.2 Boundary pseudo–differential equation

The next Proposition gives us the representation of the solution of the BVP set in D via the potential $P_{D\Gamma}$.

PROPOSITION 3.4 Assume that

$$\begin{aligned} &L_D V_D = f_D, \\ &V_D \in \Xi_D. \end{aligned} \tag{16}$$

Then, there exists a solution of (16) with $\mathrm{Tr}^+(\Gamma) V_D = \xi_\Gamma \in \pi(\Gamma)$ iff

$$\xi_\Gamma = P_\Gamma \xi_\Gamma + \mathrm{Tr}^+(\Gamma)(G_D f_D). \tag{17}$$

If equality (17) is valid, then the solution of BVP (16) having $\mathrm{Tr}^+(\Gamma) V_D = \xi_\Gamma$ is unique and given by

$$V_D = P_{D\Gamma} \xi_\Gamma + G_D f_D. \tag{18}$$

Proof. If the solution V_D exists then from the generalized Green's formula (13) it follows that equality (18) is valid. Applying the operator $\mathrm{Tr}^+(\Gamma)$ to both sides of (18), we obtain (17).

If now equality (17) is valid, then the function V_D in (18) has the following trace ξ_Γ:

$$\mathrm{Tr}^+(\Gamma) V_D = P_\Gamma \xi_\Gamma + \mathrm{Tr}^+(\Gamma)(G_D f_D) = \xi_\Gamma$$

On the other hand, the function V_D is a solution of (16). Indeed,

$$L_D V_D = L_D P_{D\Gamma} \xi_{\Gamma_+} + L_D G_D f_D = f_D.$$

It is not difficult now to prove that the function V_D with $\mathrm{Tr}^+(\Gamma) V_D = \xi_\Gamma$ such that $L_D V_D = f_D$ is unique. This immediately follows from the uniqueness of the potential $P_{D\Gamma} \xi_\Gamma$. $\Box$

Thus, equality (17) provides the necessary and sufficient condition for ξ_Γ to be extended to the interior of the domain D as a function $V_D:\ L_D V_D = f_D$ [1]. Although V_D from (18) is unique, the solution of the boundary equation (17) is not unique.

It is also important to note that relation (17) does not depend on the boundary conditions on the external boundary Γ^0 and on the structure of the operator

L in D^- provided it does not violate the assumptions of BVP (1). In particular, equality (17) does not depend on the domain D^-. Therefore, equality (17) can be interpreted as the "internal" boundary condition [1] for the subdomain D. It is worth noting that in the case of the operator L corresponding to either the Poisson equation or Helmholtz equation, equality (17) includes the Fredholm equation of the second kind for the density of the potential. In addition, equality (17) contains another equation which excludes the possibility of internal resonance [1].

As an example, let us consider the following Dirichlet BVP set in D:

$$\begin{aligned} &L_D U_D = f_D, \\ &U_{D|\Gamma} = U_\Gamma, \\ &\theta(D) f_D \in F_{D^0}. \end{aligned}$$

U_D can be represented as $U_D = U_{D,l} + U_{D,f}$ where $U_{D,f} = G_D(\theta(D)f_D)$ and $U_{D,l}$ is the solution of the following homogeneous BVP: $L_D U_{D,l} = 0_D$ and $U_{D,l|\Gamma} = U_\Gamma - U_{D,f|\Gamma}$.

Since, $P_\Gamma U_{D,f|\Gamma} = 0_\Gamma$, then equality (17) is reduced to the following equation:

$$P_\Gamma \operatorname{Tr}^+(\Gamma) U_{D,l} = \operatorname{Tr}^+(\Gamma) U_{D,l}.$$

The solution of this equation does not depend on the extension of $U_{D,l}$ outside D in view of the definition of the potential.

The numerical solution of the pseudo–differential equation (17) can be effectively realized via the DPM. In the numerical realization of the DPM the choice of the domain D^0 and the boundary conditions on Γ^0 is important since they affect the well–posedness of the so called the auxiliary problem [1]. The auxiliary problem represents a BVP in D^0 with a specifically chosen right–hand side corresponding to $\theta(\overline{D^-})LV$ in (7).

Next, from (18) we have:

$$V_D = P_{DD} V_D + G_D f_D.$$

Meanwhile, from Corollary 2 of Proposition 3.3, it follows that:

$$V_D \in \ker P_{DD} \oplus \mathrm{Im\ P_{DD}}.$$

Thereby, if $V_D \in \mathrm{Im\ P_{DD}}$, then $V_D = P_{DD} V_D$ and V_D retains if the auxiliary problem changes due to the change of either the boundary conditions or the domain D^-. In turn, if $V_D \in \ker P_{DD}$, then $V_D = G_D f_D$. Obviously, $V = G^{-1}\theta(D) f_D$ depends on the auxiliary problem. Hence, V_D depends either. Thus, only $\mathrm{Im\ P_{DD}}$ is invariant to the change of the auxiliary problem.

The finite–difference counterpart of Proposition 3.4 is one of the basic theorems in the DPM. It allows us to reduce the solution of the BVP set on some quite arbitrary domain D to the solution of the BVP formulated on some domain $D^0 :\ D \subset D^0$. Here, we can effectively exploit the fact that the domain D^0 can be as simple as we chose. Then, the numerical solution of the BVP set in D^0 can be simpler than the solution of the original BVP if the domain D is complicated. In other words, it allows us to represent Green's operator G_D via G_{D^0} which is either known or easy to find.

Along with the potential $P_{D\Gamma}$ in D, we can introduce the potential $Q_{D^-\Gamma}$ in D^- as follows.

3.3 Potential on the external subdomain

Definition 3.5 Let $V \in \Xi_{D^0}$, then an operator $Q_{D^-D^-}$ is defined as

$$Q_{D^-D^-}V_{D^-} \stackrel{def}{=} V_{D^-} - G_{D^-}L_{D^-}V_{D^-}, \tag{19}$$

where the definition of the operators G_{D^-} and L_{D^-} are similar to the definition of G_D and L_D, respectively.

It is clear that $\mathrm{Im}\ \mathrm{Q_{D^-D^-}} = \ker \mathrm{L_{D^-}}$. Similarly to $\mathrm{Tr}^+(\Gamma)$, we are able to introduce an operator $\mathrm{Tr}^-(\Gamma) : \Xi_{D^0} \to \pi(\Gamma)$ such that the pair of $(\mathrm{Tr}^-(\Gamma), \pi(\Gamma))$ creates a clear trace associated with $Q_{D^-D^-}$.

Definition 3.6 Let $V \in \Xi_{D^0}$ and $\xi_\Gamma = \mathrm{Tr}^-(\Gamma)V \in \pi(\Gamma)$. Then

$$Q_{D^-\Gamma}\xi_\Gamma \stackrel{def}{=} Q_{D^-D^-}V_{D^-},$$

In the case of the potential $Q_{D^-D^-}$ the total boundary Γ_t includes the boundary of the domain D^0: $\Gamma^0 := \partial D^0$. A clear trace of $Q_{D^-D^-}$ in space Ξ_{D^0} onto the boundary Γ is given by the pair of $(\mathrm{Tr}^-(\Gamma), \pi(\Gamma))$. Meanwhile, for example, the clear trace in space $H^s(D^0)$ does not coincide with the clear trace in Ξ_{D^0} and must be completed by the appropriate boundary condition on the boundary Γ^0. Thus, if we consider the necessary and sufficient condition for clear trace ξ_{Γ_t} to be extended to the interior of the domain D^-:

$$\xi_{\Gamma_t} = Q_{\Gamma_t}\xi_{\Gamma_t} + \mathrm{Tr}^-(\Gamma_t)(G_{D^-}f_{D^-}), \tag{20}$$

then the appropriate boundary condition on the boundary Γ^0 must be included in the clear trace.

3.4 Criterion of the clear trace

We now obtain the criterion for a clear trace and derive differential equations for the potentials P_{DD} and $Q_{D^-D^-}$. For this purpose, let us now introduce a boundary operator in D^0 as follows:

$$L_\Gamma U \stackrel{def}{=} LU - \{LU\}, \tag{21}$$

where $\{LU\}$ means the regular part of the function LU in D^0. In the case of L_f with infinitely differentiable coefficients we have:

$$\langle LU, \Phi\rangle = \langle U, L^*\Phi\rangle = \int_{D^0}(U, L^*\Phi)d\mathbf{y} = \int_{D}(U, L^*\Phi)d\mathbf{y} +$$
$$\int_{D^-}(U, L^*\Phi)d\mathbf{y} = -\int_D \sum_1^m \frac{\partial}{\partial y^i}(U, A^{i^T}\Phi)d\mathbf{y} + \int_D (LU, \Phi)d\mathbf{y} +$$
$$-\int_{D^-}\sum_1^m \frac{\partial}{\partial y^i}(U, A^{i^T}\Phi)d\mathbf{y} + \int_{D^-}(LU, \Phi)d\mathbf{y} = (\{LU\}, \Phi) +$$
$$(A_n [U]_\Gamma, \Phi).$$

From here on, (U, V) is a scalar product of vector–functions U and V, $L^* \stackrel{def}{=} -\sum_1^m A^{i^T}\frac{\partial}{\partial y^i} + B^T$, $[U]_\Gamma \stackrel{def}{=} \mathrm{Tr}_\Gamma^- U - \mathrm{Tr}_\Gamma^+ U$. Thus,

$$L_{f|\Gamma}U = A_n [U]_\Gamma \delta(\Gamma). \tag{22}$$

It is possible to prove that the properties of the coefficients determined in (5) are sufficient for equality (22). For this purpose, it is enough to rewrite $L_f U$ in the following equivalent form:

$$L_f U = \sum_1^m \frac{\partial}{\partial y^i}(A^i U) + (B - \sum_1^m \frac{\partial}{\partial y^i}A^i)U,$$
$$U \in \Xi_{D^0}.$$

In order to consider the operator L_s, let us consider

$$\nabla U = \{\nabla U\} + [U]_\Gamma \mathbf{n}\delta(\Gamma) \tag{23}$$

and the generalized Green's formula for distributions [9], [15]:

$$\Delta U = \{\Delta U\} + \left[\frac{\partial U}{\partial \mathbf{n}}\right]_\Gamma \delta(\Gamma) + \frac{\partial}{\partial \mathbf{n}}\left([U]_\Gamma \delta(\Gamma)\right).$$

Then,

$$\nabla(p\nabla U) = \{\nabla(p\nabla U)\} + p_\Gamma \left[\frac{\partial U}{\partial \mathbf{n}}\right]_\Gamma \delta(\Gamma) + \frac{\partial}{\partial \mathbf{n}}\left(p_\Gamma [U]_\Gamma \delta(\Gamma)\right).$$

Hence,

$$L_s U = \{L_s U\} + p_\Gamma \left[\frac{\partial U}{\partial \mathbf{n}}\right]_\Gamma \delta(\Gamma) + \frac{\partial}{\partial \mathbf{n}}\left(p_\Gamma [U]_\Gamma \delta(\Gamma)\right)$$

and

$$L_{s|\Gamma}U = p_\Gamma \left[\frac{\partial U}{\partial \mathbf{n}}\right]_\Gamma \delta(\Gamma) + \frac{\partial}{\partial \mathbf{n}}\left(p_\Gamma [U]_\Gamma \delta(\Gamma)\right) \tag{24}$$

From (22) and (24) it follows that if $L := L_f$, then $A_\Gamma = A_n$, and if $L := L_s$, then $A_\Gamma = p_\Gamma E$. If $L := \nabla^l$, then by a recurrent chain one can prove that

$$\nabla^l U = \sum_{i=0}^{l-1} \frac{\partial^{l-1-i} U}{\partial \mathbf{n}^{l-1-i}}_{|\Gamma} \frac{\partial^i \delta(\Gamma)}{\partial \mathbf{n}^i} + \{\nabla^l U\}$$

and $A_\Gamma = E$, where E is the unit $l \times l$ matrix.

Let us consider the general equation of higher order:

$$LU := L^{(k)} U = \sum_{i,j=1}^{m} b_{ij} \nabla_i L_j^{(k-1)} U. \tag{25}$$

Here, $L_j^{(k-1)}$ are differential operators of order $k-1$, b_{ij} are smooth enough coefficients. Then,

$$L^{(k)} U = \{L^{(k)} U\} + \zeta_\Gamma^{(k)} A_{\Gamma, L^{(k)}} \left[U\right]_{c,\Gamma},$$

where $\left[U\right]_{c,\Gamma} \overset{def}{=} \mathrm{Tr}_c^-(\Gamma)U - \mathrm{Tr}_c^+(\Gamma)U$, $\zeta_\Gamma^{(k)} = \left(\frac{\partial^{k-1}\delta(\Gamma)}{\partial \mathbf{n}^{k-1}}, ..., \delta(\Gamma)\right)$, the matrix $A_{\Gamma,L^{(k)}}$ is a lower triangular matrix with elements including differential operators on the manifold Γ. If the coefficients of the operator $L^{(k)}$ are dimensionless, then the dimension of an element (i,j) of the matrix is given by

$$\dim A_{\Gamma,L^{(k)}}^{(i,j)} = \mathrm{length}^{j-i} \quad (i \geq j),$$

which can be proven via the method of induction. Indeed, from (23) we have

$$\begin{aligned} &\nabla L_j^{(k-1)} U = \{\nabla L_j^{(k-1)} U\} + [\{L_j^{(k-1)} U\}]_\Gamma \mathbf{n} \delta(\Gamma) + \\ &\nabla(\zeta_\Gamma^{(k-1)} A_{\Gamma, L^{(k-1)}} \left[U\right]_{c,\Gamma}). \end{aligned} \tag{26}$$

In (26), the discontinuity of any derivative can be represented via the discontinuity of the normal derivatives, see e.g. [22]:

$$[\{L_j^{(k-1)} U\}]_\Gamma = \sum_{1}^{k-1} D_{j,p} \left[\frac{\partial^{k-1-p}}{\partial \mathbf{n}^{k-1-p}} U\right]_\Gamma.$$

Here, $D_{j,p}$ are the differential operators of order p on the manifold Γ.

Having considered the appropriate conormal derivatives related to the matrix b_{ij} in (25), we find that the singular part of $L^{(k)}U$ has the form of $\zeta_\Gamma^{(k)} A_{\Gamma,L^{(k)}} \left[U\right]_{c,\Gamma}$. Thus, if $U \in \Xi_{D^0}$, then $L_\Gamma U = L_\Gamma \left[U\right]_{c,\Gamma}$ and

$$L_\Gamma = \zeta_\Gamma A_\Gamma \left[U\right]_{c,\Gamma}$$

and the function ζ is given by (11). A similar statement is also valid for the operator L_{Γ_+}.

Thus, for any function U smooth enough across Γ: $\mathrm{Tr}_c^+(\Gamma)U = \mathrm{Tr}_c^-(\Gamma)U$, we have

$$L_\Gamma U = 0.$$

Let us also introduce operator $L_{\Gamma_-} \stackrel{def}{=} L\theta(D^-) - \theta(D^-)L : \Xi_{D^0} \to F_{D^0}$. Similarly to L_{Γ_+}, we can represent L_{Γ_-} as follows

$$L_{\Gamma_-}U = \zeta_\Gamma A_\Gamma \,\mathrm{Tr}_c^-(\Gamma)U.$$

In the general case the potential $P_{D\Gamma}U_\Gamma$ can be obtained as the solution of some BVP.

Proposition 3.7 If $\xi_\Gamma \in \pi_c(\Gamma)$, then

$$P_{D\Gamma}\xi_\Gamma = -G_D(L_\Gamma\xi_\Gamma) := -G_D(\zeta_\Gamma A_\Gamma \xi_\Gamma) \tag{27}$$

and the potential $P_{D\Gamma}\xi_\Gamma$ can smoothly be extended to a function W: $W \in \Xi_{D^0}$, $\operatorname{supp} W \subset D^-$.

Proof. If $\xi_\Gamma \in \pi_c(\Gamma)$, then from the trace theorem [10, Ch.1] there exists $V \in H^s(D) \cap H_0^s(D^-)$: $\mathrm{Tr}_c^+(\Gamma)V = \xi_\Gamma$. Since the potential $P_{D\Gamma}\xi_\Gamma$ does not depend on V_{D^-}, we can set $U_D = V_D$ and $U_{D^-} = 0_{D^-}$. It is clear that $U_D \in \Xi_D$. Taking into account that $\mathrm{Tr}_c^+(\Gamma)U = \xi_\Gamma$, we have

$$\langle LU, \Phi\rangle = (\{LU\}_D, \Phi) - (L_\Gamma\xi_\Gamma, \Phi).$$

for any test function Φ . Hence, $LU - \theta(D)LU = L_{\Gamma_+}U = -L_\Gamma\xi_\Gamma \in F_{D^0}$, and equality (27) is valid.

For any $\epsilon > 0$ there exits [9] a function $\eta_\epsilon \in C^\infty(\mathbb{R}^m)$ such that $0 \le \eta_\epsilon(\mathbf{x}) \le 1$, $\eta_\epsilon = 1$ on $\overline{D}$ and $\overline{D} \subset \operatorname{supp}\eta_\epsilon \subset D_\epsilon \subset D^0$ where $D_\epsilon \to D$ if $\epsilon \to 0$. Thus, there exists ϵ_0 : $W_{D^0} = \eta_{\epsilon_0}U_D \in \Xi_{D^0}$. It is easy to see that $L_D V_D = 0_D$ and $\operatorname{supp} L_{D^0}W_{D^0} \subset D^-$. $\square$

The last statement of the Proposition also follows from the trace theorem [10, Ch.1].

Similarly to Proposition 3.7, one can prove that

Proposition 3.8 If $\xi_\Gamma \in \pi_c(\Gamma)$, then

$$Q_{D^-\Gamma}\xi_\Gamma = G_{D^-}(L_\Gamma\xi_\Gamma) := G_{D^-}(\zeta_\Gamma A_\Gamma\xi_\Gamma) \tag{28}$$

and the potential $Q_{D^-\Gamma}\xi_\Gamma$ can smoothly be extended to a function W: $W \in \Xi_{D^0}$, $\operatorname{supp} W \subset D$.

The next Proposition immediately follows from Propositions 3.7 and 3.8:

Proposition 3.9 If $\xi_\Gamma \in \pi_c(\Gamma)$, then the solution of BVP

$$\begin{aligned} &LU = L_\Gamma\xi_\Gamma, \\ &U \in \Xi_{D^0} \end{aligned} \tag{29}$$

is given by

$$U = \begin{cases} -P_{D\Gamma}\xi_\Gamma, & \text{in } D \\ Q_{D^-\Gamma}\xi_\Gamma, & \text{in } D^-. \end{cases} \tag{30}$$

Let $L = L_s$ and $\xi_\Gamma \in \pi_c(\Gamma)$. If $\xi_\Gamma = (\xi_0,\ 0)^T$, then the potentials $P_{D\Gamma}\xi_\Gamma$ and $Q_{D^-\Gamma}\xi_\Gamma$ are represented by the potential of a double–layer:

$$\begin{aligned} L_s U &= \frac{\partial}{\partial \mathbf{n}}\left(p_\Gamma \xi_0 \delta(\Gamma)\right) \qquad (31)\\ U &\in \Xi_{D^0}. \end{aligned}$$

In turn, if $\xi_\Gamma = (0,\ \xi_1)^T$, then the potentials $P_{D\Gamma}\xi_\Gamma$ and $Q_{D^-\Gamma}\xi_\Gamma$ correspond to the potential of a single–layer:

$$\begin{aligned} L_s U &= p_\Gamma \xi_1 \delta(\Gamma) \qquad (32)\\ U &\in \Xi_{D^0}. \end{aligned}$$

In both cases $P_{D\Gamma}\xi_\Gamma = -U_D$ and $Q_{D^-\Gamma}\xi_\Gamma = U_{D^-}$.

In some simple cases, we can obtain the relation between the classical potentials and the Calderon–Ryaben'kii potentials. For example, having considered the Laplace operator instead of L_s in BVP (31), its solution is given by the potential of a double layer [9]:

$$P_{D\Gamma}\xi_\Gamma = \int_\Gamma \xi_0 \frac{\partial Gr}{\partial \mathbf{n}} d\sigma,$$

where Gr is the Green's function, the surface integral represents the appropriate convolution.

Taking into account the uniform limit of the double–layer potential on the boundary [9], we arrive at the following Fredholm equation of second kind:

$$\mathrm{Tr}_\Gamma^+ P_{DD} V_D = \frac{\xi_0}{2} + \int_\Gamma \xi_0 \frac{\partial Gr}{\partial \mathbf{n}} d\sigma. \tag{33}$$

where $V_D \in \Xi_D$. Equation (33) determines an operator T: $V_D \to \xi_0$. One can prove [1] that the operator T corresponds to the clear trace. Indeed, ξ_0 is equal to zero only if the potential $P_{DD}V_D$ equals zero. The clear trace operator represented by T, obviously, is nonlocal.

The next proposition is important for the analysis of the discontinuities of the generalized Calderón–Ryaben'kii potentials across the boundary Γ.

Proposition 3.10 Let us consider the following BVP in D^0

$$\begin{aligned} LU &= L_\Gamma \xi_\Gamma, \\ U &\in \Xi_{D^0}. \end{aligned}$$

Assume that $\xi_\Gamma \in \pi_c(\Gamma)$. Then,

$$P_\Gamma \, \mathrm{Tr}_c^-(\Gamma) U = 0_\Gamma, \tag{34}$$

$$Q_\Gamma \, \mathrm{Tr}_c^+(\Gamma) U = 0_\Gamma. \tag{35}$$

Proof. Since the potential $P_{D\Gamma}\operatorname{Tr}^-(\Gamma)U$ does not depend on the extension to D^- we can set $V_{\overline{D}^-}=U_{\overline{D}^-}$ in Definition 2.5 of the potential $P_{D\Gamma}$. The function $V_{\overline{D}^-}$ can smoothly be extended to D^+: $\operatorname{Tr}^+(\Gamma)V=\operatorname{Tr}^-(\Gamma)U$. Thus, $\operatorname{supp} LV\subset D^+$. Hence, $P_{D\Gamma}\operatorname{Tr}_c^-(\Gamma)U=0_D$. Similarly, one can prove (35). □

Corollary. $P_\Gamma Q_\Gamma\xi_\Gamma=Q_\Gamma P_\Gamma\xi_\Gamma=0_\Gamma$.

Now, we can prove the criterion of a clear trace

PROPOSITION 3.11 The pair of $(\xi_\Gamma,\pi_c(\Gamma))$ is a clear trace iff $\zeta_\Gamma A_\Gamma\xi_\Gamma\in F_{D^0}$.

Proof. In one implication this statement follows from Propositions 3.7 and 3.8. To prove this statement in the opposite implication, let us consider the following BVP:

$$\begin{aligned}&LU=\zeta_\Gamma A_\Gamma\xi_\Gamma,\\&U\in\Xi_{D^0}.\end{aligned}$$

Function $V^+=\theta(D)U$ is the solution of BVP

$$\begin{aligned}&LV^+=-\zeta_\Gamma A_\Gamma\operatorname{Tr}_c^+(\Gamma)U,\\&V^+\in\Xi_{D^0},\end{aligned}$$

while function $V^-=\theta(D^-)U$ is the solution of BVP

$$\begin{aligned}&LV^-=\zeta_\Gamma A_\Gamma\operatorname{Tr}_c^-(\Gamma)U,\\&V^-\in\Xi_{D^0},\end{aligned}$$

and $\xi_\Gamma=[U]_{c,\Gamma}$. Hence, $V_D^+=P_{D\Gamma}\operatorname{Tr}_c^+(\Gamma)U$ while $V_D^-=Q_{D^-\Gamma}\operatorname{Tr}_c^-(\Gamma)U$. From Propositions 3.7 and 3.8, there exist functions $W^+\in\Xi_{D^0}$ and $W^-\in\Xi_{D^0}$: $W_D^+=V_D^+$, $\operatorname{supp}W^+\subset D^-$ and $W_{D^-}^-=V_{D^-}^-$, $\operatorname{supp}W^-\subset D^+$.

Let us now consider the function: $\widetilde{U}=W^--W^+\in\Xi_{D^0}$, and $\operatorname{Tr}_c^+(\Gamma)\widetilde{U}=\operatorname{Tr}_c^+(\Gamma)W^--\operatorname{Tr}_c^+(\Gamma)W^+=\xi_\Gamma$. Then, $P_{D\Gamma}\xi_\Gamma=-W_D^+=-U_D$. Similarly, $Q_{D^-\Gamma}\xi_\Gamma=W_{D^-}^-=U_{D^-}$.

Thus, the pair of $(\xi_\Gamma,\pi_c(\Gamma))$ is the clear trace of the potentials P_{DD} and $Q_{D^-D^-}$. □

Following [1], we can combine the potentials $P_{D\Gamma}$ and $Q_{D^-\Gamma}$ into the Cauchy–type operator [1] $R_{D^0\Gamma}:\pi(\Gamma)\to\Xi_{D^0}$ with density $\xi_\Gamma\in\pi(\Gamma)$ as follows:

$$R_{D^0\Gamma}\xi_\Gamma\overset{def}{=}\begin{cases}-P_{D\Gamma}\xi_\Gamma \text{ on } D,\\ Q_{D^-\Gamma}\xi_\Gamma \text{ on } D^-.\end{cases}\tag{36}$$

The pair $([V]_{c,\Gamma},\pi_c(\Gamma))$ creates a clear trace associated with the operator $R_{D^0\Gamma}$ and $R_{D^0\Gamma}\xi_\Gamma\in\Xi_{D^0}$. This statement follows from Definition 2.3, the properties of the potentials $P_{D\Gamma}$ and $Q_{D^-\Gamma}$, and the uniqueness of the solution of the BVP (29).

By immediate substitution one can prove that the solution of (29) is represented by

$$U=V-G\left\{LV\right\}_{D^0},\tag{37}$$

where $V \in \Xi_{D^0}$ and $[V]_{c,\Gamma} = \xi_\Gamma$. From (30), (36) and (37) it can be shown that for any $V \in \Xi_{D^0}$ the following equality is valid:

$$V_{D^0} = R_{D^0\Gamma}\,[V]_{c,\Gamma} + G\,\{LV\}_{D^0}\,. \tag{38}$$

Thus, we can decompose the space Ξ_{D^0} as follows: $\Xi_{D^0} = \Xi_{c,D^0} \oplus \Xi_{d,D^0}$. Here, Ξ_{c,D^0} is the space of continuous functions across Γ, Ξ_{d,D^0} is the space of discontinuous functions satisfying the homogeneous equation on both domains D and D^-.

From (38), it follows that if a function $V \in \Xi_{D^0}$ has a discontinuity on Γ: $[V]_{c,\Gamma} = \xi_\Gamma$, and V satisfies the homogeneous equation on both D and D^-, then it is uniquely recovered via the potentials $P_{D\Gamma}\xi_\Gamma$ and $Q_{D^-\Gamma}\xi_\Gamma$. Thus, there is a deep analogy between these potentials and the Cauchy type integral in complex analysis.

The following proposition gives a relation between P_Γ and Q_Γ.

PROPOSITION 3.12 Let the pair of $(\xi_\Gamma, \pi_c(\Gamma))$ be a clear trace. Then,

$$P_\Gamma\xi_\Gamma + Q_\Gamma\xi_\Gamma = \xi_\Gamma. \tag{39}$$

Proof. It immediately follows from Propositions 3.9 and 3.11 if we set $U = G_{D^0}L_\Gamma\xi_\Gamma$. □

Thus, the space of the Cauchy data of continuous functions U: $[U]_{c,\Gamma} = 0_\Gamma, U \in \Xi_{D^0}$ is decomposed into a direct sum of clear traces of functions satisfying the homogeneous equation (1) on either domain D or domain D^-. This result was proved by Seely [4] for elliptic equations. In the general case of discontinuous functions from Ξ_{D^0}, this statement is not valid.

It appears that the boundary equality (39) can be extended to any function $V \in \Xi$ if we set $\xi_\Gamma = [V]_{c,\Gamma}$. Indeed, from (38) it follows that $V = U + W$, where $U = R_{D^0\Gamma}\xi_\Gamma$ and $W = G\,\{LV\}_{D^0}$. Then, $\xi_\Gamma = [U]_{c,\Gamma}$, and equality (39) immediately follows from Proposition 3.10 and the following equalities:

$$P_\Gamma\xi_\Gamma = -\operatorname{Tr}_c^+(\Gamma)U,$$
$$Q_\Gamma\xi_\Gamma = \operatorname{Tr}_c^-(\Gamma)U.$$

Let us next introduce the generalized Cauchy data on Γ:

$$\operatorname{Tr}_c(\Gamma)V \overset{def}{=} \frac{1}{2}(\operatorname{Tr}_c^+(\Gamma)V + \operatorname{Tr}_c^-(\Gamma)V).$$

Then, equality (39) is also valid for $\xi_\Gamma = \operatorname{Tr}_c(\Gamma)V$.

In this case, following the previous proof we consider $V = U + W$. Then, equality (39) is obtained from

$$P_\Gamma \operatorname{Tr}_c(\Gamma)U = \frac{1}{2}\operatorname{Tr}_c^+(\Gamma)U,$$
$$Q_\Gamma \operatorname{Tr}_c(\Gamma)U = \frac{1}{2}\operatorname{Tr}_c^-(\Gamma)U,$$

and Proposition 3.12.

Now, let us consider some applications of the Calderón–Ryaben'kii potentials.

4 Active noise shielding problem

Suppose that problem (1), (2) describes an acoustic field in the domain D^0. The sources situated in D are considered as wanted, while those situated outside D are interpreted as unwanted sources of noise.

Assume that we know the value of the function U in some neighborhood of the boundary Γ. We note that only this information is assumed to be available. In particular, the distribution of the sources $f := F$ on the right-hand side of the BVP is unknown. The AS problem is reduced to searching additional sources g on $\overline{D^-}$ such that the solution of the BVP

$$\begin{aligned} &LU^{(g)} = F + g, \qquad (40)\\ &F \in F_{D^0},\\ &\text{supp } g \subset \overline{D}^-,\\ &U^{(g)} \in \Xi_{D^0} \end{aligned}$$

coincides on D with the solution U of BVP (1), (2) if $F = f^+$. An "obvious" solution $F = -f^-$ is not applicable here because the distribution of f^- is unknown. The solution of this problem can be derived via the generalized potentials as follows.

Let us introduce the following two BVPs:

$$\begin{aligned} &LU^+ = f^+, \qquad (41)\\ &\text{supp } f^+ \subset D,\\ &U^+ \in \Xi_{D^0}, \qquad (42) \end{aligned}$$

and

$$\begin{aligned} &LU^- = f^-, \qquad (43)\\ &\text{supp } f^- \subset D^-,\\ &U^- \in \Xi_{D^0}. \qquad (44) \end{aligned}$$

From Proposition 3.2, it follows that the requirements of the noise cancelation is equivalent to

$$P_{DD}L_D^{-1}(f+g) = P_{DD}U_D^+ = 0_D$$

On the other hand, from Proposition 3.7 and Proposition 3.3, we have:

$$P_{DD}L_D^{-1}f = P_{D\Gamma}\xi_\Gamma = L_D^{-1}(-\zeta_\Gamma A_\Gamma \xi_\Gamma),$$

where $\xi_\Gamma = \mathrm{Tr}_c(\Gamma)U$, and

$$P_{DD}L_D^{-1}g = L_D^{-1}g.$$

Thus, we can choose

$$g = g_0 + LW, \tag{45}$$

$$g_0 \overset{def}{=} \zeta_\Gamma A_\Gamma \xi_\Gamma, \tag{46}$$

where W is any function from Ξ_{D^0} such that $\operatorname{supp} W \subset D^-$.

It can be shown that the solution of BVP

$$LV = g_0, \tag{47}$$
$$V \in \Xi_{D^0}$$

is the following:

$$L^{-1} g_0 = \begin{cases} -U^-, & \text{in} \ \ D \\ U^+, & \text{in} \ \ D^-. \end{cases} \tag{48}$$

The term g_0 represents the surface potential part of the AS solution [19]. In the application to the differential operators L_f and L_s, we obtain the following AS source terms:

$$g_{0|f} = A_n U_\Gamma \delta(\Gamma)$$

and

$$g_{0|s} = p_\Gamma \frac{\partial U}{\partial \mathbf{n}}_{|\Gamma} \delta(\Gamma) + \frac{\partial}{\partial \mathbf{n}} \left(p_\Gamma U_\Gamma \delta(\Gamma) \right).$$

In particular cases of the Euler acoustics equations and Helmholtz equation the AS solution (46) provides the source terms obtained in [19] and [8], respectively.

The solution (46) is general and can be applied to different kind of the operator L; for example, it can be used for the Maxwell equations [21].

Let us now analyze the solution of BVP (40) with g determined by (45), (46). The realization of the source (46) is based on the knowledge (measurement) of $\mathrm{Tr}_c(\Gamma)U$. Once the AS source is implemented, the field changes in the shielded domain D and, possibly, outside. Moreover, the field $U^{(g)}$ becomes discontinuous across the boundary Γ. In the domain D we have $\mathrm{Tr}_c^+(\Gamma)U^{(g)} = \mathrm{Tr}_c(\Gamma)U^+$. Thus, the measured field coincides with the case $f^- \equiv 0$ and the AS is not required. Hence, the implementation of the AS source leads to some uncertainty. This fact can especially be important if the field f^- changes in time.

In the external domain D^-, the field corresponding to $\mathrm{Tr}_c^-(\Gamma)U^{(g)}$ may also change in comparison to U_Γ due to the additional field generated by the secondary source g if $f^+ \neq 0$, in particular. On the other hand, from Proposition 3.10 we have $P_\Gamma \, \mathrm{Tr}_c^-(\Gamma)U^{(g)} = \mathrm{Tr}_c(\Gamma)U^-$. Thus, the potential $P_\Gamma \, \mathrm{Tr}_c^-(\Gamma)U^{(g)}$ filters the contribution of the secondary term g_0, and the value of the AS source term is given by

$$g_0 = \zeta_\Gamma A_\Gamma P_\Gamma \, \mathrm{Tr}_c^-(\Gamma)U^{(g)}. \tag{49}$$

Hence, the measurements must be performed at the external boundary and the realization of AS requires the solution of a BVP in the domain D^0. We note that the AS (49) gives an optimal solution because $P_\Gamma \operatorname{Tr}_c(\Gamma)U^+ = 0_\Gamma$. Thus, this solution efficiently filters the "friendly" sound which does not require to be shielded. From (47) and (48), it follows that the secondary source does not affect the field outside $\overline{D}$ if $f^+ \equiv 0$. Then, the solution of the additional BVP is not required since $P_\Gamma \operatorname{Tr}_c^-(\Gamma)U^{(g)} = \operatorname{Tr}_c^-(\Gamma)U_\Gamma$ and the right–hand side is assumed to be immediately obtained from the measurements.

The developed AS solution can immediately be applied to nonstationary problem with some minor modifications in the theory.

5 Artificial boundary conditions

Assume BVP (1), (2) holds such that the source terms are only situated on D: $\operatorname{supp} f \subset D$. It is possible to exactly transfer the boundary conditions from the boundary Γ^0 to the boundary Γ. In the continuous space they are formulated as follows. Having applied Proposition 3.4 to the domain D^-, we find that the vector–function

$$\xi_\Gamma = \operatorname{Tr}_c^-(\Gamma)U \in \pi_c(\Gamma) \subset \oplus_0^{k-1} H^{s-1/2-j}(\Gamma)$$

can be extended to $U_D \in \Xi_D$ if and only if

$$Q_\Gamma \xi_\Gamma = \xi_\Gamma. \tag{50}$$

Thus, the condition (50) determines the subspace $\pi_c^+(\Gamma)$ of the boundary vector–functions from $\pi_c(\Gamma)$ to be the trace $\operatorname{Tr}_c^+(\Gamma)U_D$ of the solution of BVP (16).

Thus, it is possible to say that the boundary condition on the original boundary Γ^0 is exactly transferred to the boundary Γ via the condition (50). It is clear that this boundary condition is not local. It can be reformulated in the form of a pseudo–differential boundary equation

$$\frac{\partial U}{\partial \mathbf{n}}_{|\Gamma} = R_\Gamma U_\Gamma, \tag{51}$$

where R_Γ is a nonlocal operator of Poincaré–Steklov type. The described approach is used in [23] to develop nonlocal wall–functions for turbulence modeling. Another particular class of the Poincaré–Steklov operators are provided by the Dirichlet-to-Neumann (DtN) maps [24].

In turn, it is possible to transfer the boundary conditions from the boundary Γ (if they are set there) to a remote artificial boundary Γ_0. This makes sense if the domain D is complex while the domain D^0 is "simple".

For this purpose, let us consider a uniquely solvable and well–posed BVP, which is formulated on $D \subset D^0$:

$$LU = f,$$
$$l_\Gamma U = \alpha_\Gamma,$$

where l_Γ is some differential operator on the boundary Γ.

Assume that ξ_Γ is the solution of the following set:

$$\begin{aligned} Q_\Gamma \xi_\Gamma &= \xi_\Gamma, \\ \widehat{l}_\Gamma \xi_\Gamma &= \alpha_\Gamma. \end{aligned} \tag{52}$$

The first equation in (52) determines the subspace $\pi_c^+(\Gamma)$, while the second equation restricts it to the traces of the functions satisfying the boundary conditions. Then, the solution of the following BVP formulated on D^0

$$\begin{aligned} &LW = f - \zeta_\Gamma A_\Gamma \xi_\Gamma, \\ &W \in \Xi_{D^0} \end{aligned} \tag{53}$$

is given by $W = \theta(D)U_D$. It immediately follows from the previous section since $-\zeta_\Gamma A_\Gamma \xi_\Gamma$ provides the AS of the domain D^- from the field generated by f.

6 Conclusion

The general theory of the Calderón–Ryaben'kii potentials has been considered via the theory of distributions. The theory allows us to reduce a uniquely solvable and well–posed linear BVP to a boundary pseudo–differential equation. The DPM provides an efficient way for the numerical solution of the boundary equation. The definition of the Calderón–Ryaben'kii potentials is based on the notion of a clear trace. The criterion of the clear trace has been formulated. On the basis of the Calderón–Ryaben'kii potential theory, the solution of the active shielding problem has been obtained in a general formulation. For the first time, the AS solution takes into account the diffraction effects such as the feedback of the AS on the input (measurement) data. It has been shown that the Calderón–Ryaben'kii potentials provide an efficient approach for the exact transfer of boundary conditions from the original boundary to an artificial boundary.

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References

[1] Ryaben'kii, V. S., 2002, *Method of difference potentials and its applications*, Berlin, Springer-Verlag.

[2] Brebbia, C. A., Telles, J. C. F., and Wrobel, L. C., 1984, *Boundary Element Techniques*, Springer–Verlag.

[3] Calderón, A. P., 1963, *Boundary–value problems for elliptic equations*, in Proceedings of the Soviet–American Conference on Partial Differential Equations at Novosibirsk, Moscow, Fizmatgiz, pp.303–304.

[4] Seeley, R. T., 1966, *Singular integrals. American Journal of Mathematics*, 88 (4), pp.781–809.

[5] Hörmander, L., 1966, Pseudo-differential operators and non-elliptic boundary problems, Ann. of Math. 83, pp. 129–209.

[6] Mikhlin, S.G., Morozov, N.F., and Paukshto, M.V., 1995, *The integral equations of the Theory of Elasticity*, B.G. Teubner Verlagsgesellschaft, Stuttgart.

[7] Kamenetskii, D.S., 2000, *On one form of representing the difference potentials*, J. Appl. Numer. Math., 33, pp.501–508.

[8] Lončarić, J., Ryaben'kii, V. S., and Tsynkov, S. V., 2001, *Active shielding and control of noise*, SIAM J. Appl. Math., 62, pp.563–596.

[9] Vladimirov, V. S., 1971, *Equations of Mathematical Physics*, Dekker, New York.

[10] Lions, J-L, and Magenes, E., 1972, *Non-homogeneous boundary value problems and applications*, Springer, Berlin-Heidelberg-New York.

[11] Nelson, P. A., and Elliott, S. J., 1992, *Active control of sound*, Academic Press, San Diego, CA, USA.

[12] Tochi, O., and Veres, S., 2002, *Active sound and vibration control. Theory and applications*, The Institution of Electrical Engineers.

[13] Ryaben'kii, V. S., 1995, *A difference shielding problem. Functional Analysis and Applications*, 29, pp.70–71.

[14] Lončarić, J., and Tsynkov, S. V., 2003, *Optimization of acoustic sources strength in the problems of active noise control*, SIAM J. Appl. Math., 63, pp. 1141–1183.

[15] Tsynkov, S. V., 2003, *On the definition of surface potentials for finite-difference operators*, J. of Scientific Computing, 18, pp.155–189.

[16] Ryaben'kii, V. S., Tsynkov, S. V., and Utyuzhnikov, S. V., 2007, *Inverse source problem and active shielding for composite domains*, Applied Mathematics Letters, 20 (5), pp. 511–515.

[17] Ryaben'kii, V. S., and Utyuzhnikov, S. V., 2006, *Active shielding model for hyperbolic equations*, IMA Journal of Applied Mathematics, 71 (6), pp. 924–939.

[18] Ryaben'kii, V. S., Utyuzhnikov, S. V., and Turan, A., 2008, *On the application of difference potential theory to active noise control*, J. Advances in Applied Mathematics, 40 (2), pp. 194–211.

[19] Ryaben'kii, V. S., and Utyuzhnikov, S. V., 2007, *Differential and finite-difference problems of active shielding*, Applied Numerical Mathematics, 57 (4), pp. 374–382.

[20] Tsynkov, S. V., 1998, *Numerical solution of problems on unbounded domains. A review*, J. of Applied and Numerical Mathematics, 27, pp. 465–532.

[21] Utyuzhnikov, S. V., 2007, *Nonstationary problem of active sound control in bounded domains*, Proceedings of Waves2007, The 8th International Conference on Mathematical and Numerical Aspects of Waves, 2007, 23–27 July 2007, pp. 443–445.

[22] Egorov, Yu.V., Shubin, M.A., 1992, *Foundations of the classical theory of partial differential equations*, Springer, Berlin–London.

[23] Utyuzhnikov, S. V., 2008, *Robin–type Wall Functions and Their Numerical Implementation*, J. Appl. Numer. Math., (to appear).

[24] Givoli, D., and Patlashenko, I., 2004, *Dirichlet–to–Newmann boundary conditions for time–dependent dispersive waves in three–dimensional guides*, J. Comput. Phys., 199, pp. 339–354.