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Active wave control and generalized surface potentials

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ABSTRACT

In active wave control, an arbitrary bounded domain with the smooth boundary is shielded from the outside field (noise) using additional sources. Unlike passive control, there is no any mechanical insulation in the system. The general solution of the problem is obtained in unsteady linear formulation. For this purpose, the theory of potentials introduced by Ryaben'kii is extended to initial–boundary value problems and the theory of distributions. Both first- and second-order spatial differentiation operators are considered. The obtained results can immediately be applied to active control problems in electromagnetics and acoustics. Two classical problems, on a bounded conductor in an electrostatic field and superconductor in a magnetostatic field, are interpreted as active control problems. The control sources for aeroacoustics are then obtained in the form of a linear combination of single- and double-layer sources. The constructed solution of the problem requires only the knowledge of the total field on the perimeter of the shielded domain.

1. Introduction

The paper deals with the active control (AC) problem of shielding a bounded domain from the field generated outside. All possible internal sound sources are interpreted as "friendly" whereas external sources are considered to be "noise" sources. Furthermore, only the total sound field nearby the boundary of the protected domain is assumed to be known. In the framework of the active control methods shielding of the domain is carried out by introducing additional sources so that the total (sound) field consisting of both primary and secondary sources provides the desirable shielding effect. This approach is distinctly different from the passive control methods in which the domain has

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to be mechanically insulated. Thus, the active control problem can mathematically be formulated as an inverse source problem and is immediately related to the problems of active noise shielding (see, e.g., [2,12,23]) and active vibration control [3,21].

In the current formulation the AC problem was first considered by Malyuzhinets [10] in relation to wave propagation equation. The solvability of the problem was demonstrated for unbounded spaces using the fundamental solution of the Helmholtz equation. The inverse source stationary problem was first solved in a finite-difference general formulation by Ryaben'kii in [14] using the apparatus of the Difference Potential Method (DPM) [15]. The obtained solution requires only the knowledge of the total field (containing both "friendly" and noise components) at the computational (grid) boundary of the protected domain. The general solution [14] was applied to the Helmholtz equation in [9]. It was shown via the theory of distributions that in continuous spaces the solution [14] can be represented as a linear combination of single- and double-layer sources. The optimization of the solution was studied in [6–8]. In [18] the AC problem in composite domains was formulated for the first time. Its general solution was constructed for finite-difference spaces in [18] and for continuous spaces in [13]. The principal novelty of the problem, considered in [18] and [13], was that it allowed selective communication between different sub-domains.

For the acoustics Euler equations in continuous spaces, the AC solution was first constructed in [17]. It was derived using the apparatus of distributions for time-harmonic waves under rather general assumptions. The DPM-based discrete solution was shown to approximate the continuous solution as the spatial mesh is refined. In [16], the DPM-based solution was extended to rather broad range of hyperbolic systems of equations including acoustic equations with constant and variable coefficients. It appears that in bounded domains the control sources do not disturb even the echo of the "friendly" sound component [19]. The detailed mechanism of the active noise shielding based on the solution [17] was revealed in [19].

The DPM became a powerful mathematical tool for solving complicated problems of mathematical physics, see e.g. [15]. It is based on the theory of potentials introduced by Ryaben'kii. The theory has mostly been developed in a finite-difference framework although in [15] the foundations of the theory are also given in the differential classical form. General aspects of the theory in continuous and discrete spaces are addressed in [11]. Some extension of the DPM formalism to the linear Helmholtz-type equations with discontinuous solutions can be found in [9], where it is applied to the active sound control problem. In [15], it is proven by Kamenetskii that the potentials, introduced by Calderón [1] for elliptic equations and developed by Seeley [20], are equivalent to some forms of the Ryaben'kii's potential. It is to be noted that for long time the theory of the DPM was developed by Ryaben'kii et al. completely independently from the theory of the Calderón potentials.

In the current paper, the theory of the Calderón–Ryaben'kii potentials is extended to nonstationary problems and the weak solutions using the theory of distributions (see e.g. [5,24]). This extension allows one to apply the theory to initial–boundary value problems with discontinuous solutions. Here by the Calderón–Ryaben'kii potentials we understand the extension of the Ryaben'kii difference potentials to continuous spaces. It is to be noted that the used name for the potentials is not traditional in the literature. Meanwhile, we believe that such a name is the most appropriate one. The general solution of the nonstationary AC problem is then obtained via the nonstationary potentials.

The application of the extended theory is illustrated by the examples of the Maxwell equations, wave equation and linearized Euler equations (LEE) for aeroacoustics are considered. In the case of the Maxwell equation the well-known problems of a bounded conductor in an electrostatic field and a superconductor in a magnetic field are interpreted as AC problems. The AC source terms for acoustics and aeroacoustics are obtained. It is shown that for aeroacoustics the source terms must take into account the mean flow through the boundary. In all examples the solution of the appropriate problem is derived as a particular case from the obtained general solution of the AC problem.

2. General statement of the AC problem

We consider some bounded domain D: $\overline{D} \subseteq \mathbb{R}^m$ with smooth boundary Γ_0 and a sub-domain D^+: $\overline{D^+} \subset D$ with smooth enough boundary Γ.

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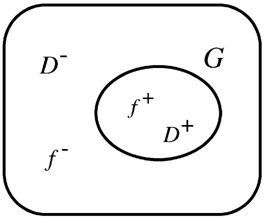

Fig. 1. Domain sketch.

Suppose that the field U is described by the following linear boundary value problem (BVP) formulated in either $\mathbb{R}^m$ or $\mathbb{R}^{m+1}$:

$$LU = f, \tag{1}$$

$$U \in \Xi_D. \tag{2}$$

Here, the operator L is a linear differential operator, Ξ_D is the functional space specified below. In particular, the operator L acting in $\mathbb{R}^{m+1}$ may correspond to the acoustic equations. We assume that BVP (1), (2) is well-posed in the sense of Hadamard for any right-hand side f: $f \in L_2^{loc}(D)$. The boundary conditions are supposed to be local and implicitly included in the definition of the space Ξ_D. It follows from the linearity of the problem and its well-posedness that the solution of the homogeneous problem (1), (2) can be only trivial. In addition, we suppose the space Ξ_D should not be degenerate. Thus, we assume that the boundary and initial conditions are not over-determined.

The sources on the right-hand side are assumed to be placed both on D^+ and outside D^+ (see Fig. 1):

$$f = f^+ + f^-,$$

$$\operatorname{supp} f^+ \subset D^+,$$

$$\operatorname{supp} f^- \subset D^- \stackrel{def}{=} D \setminus \overline{D^+}. \tag{3}$$

We interpret f^+ as "friendly" field sources, while f^- is considered as an "adverse" field (noise) sources.

Our key assumption on the input data of the AC problem is that the field U is only known in the vicinity of the boundary Γ and that any other information on either the adverse field or the boundary conditions is not available to us. Then, we arrive at the following inverse source problem: find additional sources G in $\overline{D^-}$ such that the solution of BVP

$$LW = f + G,$$

$$\operatorname{supp} G \subset \overline{D^-},$$

$$W \in \Xi_D \tag{4}$$

coincides on the domain D^+ with the solution of BVP (1), (2) if $f^- \equiv 0$:

$$LU^+ = f^+,$$

$$U^+ \in \Xi_D. \tag{5}$$

Thus, it is required that the functions U and W are identical in the domain D^+: $W_{D^+} = U^+_{D^+}$.

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Further, we specify the problem (1), (2) as follows. The function U is supposed to be the solution of a well-posed initial–boundary value problem (IBVP) in the cylinder $K_T = D \times (0,T) \subseteq \mathbb{R}^{m+1}$ $(T > 0)$:

$$LU \overset{def}{=} L_t^{(p_t)} U + L_y U = f,$$
$$U \in \Xi_D. \tag{6}$$

Here, $U \in \mathbb{R}^n$, $f \in \mathbb{R}^n$, $L_t^{(p_t)} \overset{def}{=} A^0 \frac{\partial^{p_t}}{\partial t^{p_t}}$ is a differential operator with respect to the time variable t, A^0 is an $n \times n$ matrix. The value of p_t has to satisfy the requirement of the well-posedness of the problem.

Next, we consider first- and second-order spatial differential operators L_y of general type.

The first-order operator L_y is given by

$$L_y := L_f \overset{def}{=} \sum_1^m A^i \frac{\partial}{\partial y^i} + B, \tag{7}$$

where $\{y^i\}$ $(i = 1, \ldots, m)$ is some Cartesian coordinate system; A^i, B are $n \times n$ matrices: $A^i = A^i(\mathbf{y}) \in C^1(\overline{D})$, $B = B(\mathbf{y}) \in C(\overline{D})$.

The second-order operator is given by the following elliptic operator:

$$L_y := L_s \overset{def}{=} -\nabla(p\nabla) - q, \tag{8}$$

where $p \in C^1(\overline{D})$, $q \in C(\overline{D})$ and $p > 0$.

Suppose that space Ξ_D includes functions which are smooth enough with respect to the variable t and satisfy homogeneous initial conditions. That is, if $U \in \Xi_D$, then

$$\frac{d^k}{dt^k} U(\mathbf{x}, 0) = 0 \quad (k = 0, \ldots, p_t - 1). \tag{9}$$

Let us say that a function U is a generalized solution of BVP (1), (2) if $\langle LU, \Phi \rangle = \langle f, \Phi \rangle$ for any function from some space of test functions. Here, $\langle f, \Phi \rangle$ denotes a linear continuous functional associated with the given generalized function (distribution) f. Along with a generalized function f, we introduce f_{D^+} as the restriction of f to D^+ [24].

If the right-hand side in (6) is a regular function, then the IBVP (6), (2) for finding the weak solution is reduced to the following requirement:

$$\int_0^T \int_D (LU - f, \Phi)\, d\mathbf{x}\, dt = 0$$

for any basic (test) function $\Phi \in C_0^\infty(K_T)$, where $(\cdot,\cdot)$ means a scalar product.

It is to be noted that the specifications set on BVP (1), (2) are in general sufficient for further analysis. However, we introduce some additional conditions to make this analysis more specific. The functional space Ξ_D is defined in such a way that the weak solution of IBVP (6), (2), (9) satisfies the governing equation in the classical sense almost everywhere, and it is bounded. Thus, we assume that for any $0 < t < T$: $\Xi_D \subset H^s(D \setminus \Gamma) := H^s(D^+) \cap H_0^s(D^-)$, H_0^s and H^s are Sobolev spaces, $s > k - 1/2$, $s \neq \text{integer} + 1/2$, k is the order of the operator L_y. Then, the condition of the well-posedness implies the following estimate to be valid:

$$\|U_{D^+}\|^2_{H^s} + \|U_{D^-}\|^2_{H^s} < C\big(\|LU_{|D^+}\|^2_{H^{s-k}} + \|LU_{|D^-}\|^2_{H^{s-k}}\big),$$

where $C = C(T)$ is some positive constant.

Next, suppose that in (1) the right-hand side $f \in F_D$ where the space F_D such that the solution of BVP (1), (2) exists. Thus, spaces Ξ_D and F_D are isomorphic to each other. Hereafter, we assume: $f \in F_D \Rightarrow \theta(D^+)f \in F_D$, where $\theta(D^+)$ is the characteristic function of D^+ is equal to 1 on D^+ and 0 outside. Obviously, this assumption is valid on default for the AC problem since $f^+ \in F_D$.

The solution of the stationary problem can be interpreted as the limiting stationary solution of the appropriate IBVP. Formally, this corresponds to the assumption on $A^0 \equiv 0$ and time independence.

The general solution of the AC problem, formulated above, is based on the theory of potentials described in the next section. The introduced potentials can in general be considered as an extension of the Ryaben'kii potentials to nonstationary problems and weak solutions.

3. Calderón–Ryaben'kii potentials for IBVPs

Let us first introduce in $\mathbb{R}^{m+1}$ an operator $P_{D^+} : \Xi_{D^+} \to \Xi_{D^+}$, $\Xi_{D^+} = \{U_{D^+} \mid U \in \Xi_D\}$, as follows.

Definition 1.

$$P_{D^+} V_{D^+}(\mathbf{x}, t) \overset{def}{=} V_{D^+} - \int_T \int_{D^+} Gr(\mathbf{x}|\mathbf{y}, t|\tau) LV(\mathbf{y}, \tau)\, d\mathbf{y}\, d\tau.$$

Here, Gr is Green's function of the linear BVP (1), (2), (6), (9).

In the stationary case this definition coincides with the definition of the potential introduced by Ryaben'kii [15].

Alternatively, one can introduce the following definition of the operator P_{D^+} via the theory of distributions:

Definition 2.

$$P_{D^+} V_{D^+} \overset{def}{=} L^{-1}_{D^+}\big(\theta(\overline{D^-}) LV\big), \tag{10}$$

where $L^{-1}_{D^+} g \overset{def}{=} L^{-1} g_{|D^+}$, $\theta(\overline{D^-})$ is the characteristic function of $\overline{D^-}$.

It is worth noting that since the IBVP is well-posed the inverse operator L^{-1} in Definition 2 is defined.

Note that Definition 2 follows from Definition 1. Indeed, from Definition 1

$$P_{D^+} V_{D^+} = \big(L^{-1} LV - L^{-1}\big(\theta(D^+) LV\big)\big)_{|D^+} = L^{-1}_{D^+}\big(\theta(\overline{D^-}) LV\big).$$

For further consideration, Definition 2 is more useful because it does not utilize Green's function.

The authors of [15] introduced the notion of a clear trace $\mathrm{Tr}(\Gamma) U_D$, assigned to the operator P_{D^+}, which can be defined as

$$\mathrm{Tr}(\Gamma) V_{D^+} = \mathrm{Tr}(\Gamma) W_{D^+} \quad \Rightarrow \quad P_{D^+} V_{D^+} = P_{D^+} W_{D^+}. \tag{11}$$

Here, $\mathrm{Tr}(\Gamma)$ is a boundary operator: $\Xi_{D^+} \to \Xi_\Gamma$, $\Xi_\Gamma \subset \bigoplus_{j=0}^{k-1} H^{s-1/2-j}(\Gamma)$, where k is the order of the operator L_y.

Then, we arrive at the definition of a surface potential $P_{D^+\Gamma} : \Xi_\Gamma \to \Xi_{D^+}$ with density ξ_Γ.

Definition 3.

$$P_{D^+\Gamma} \xi_\Gamma \overset{def}{=} P_{D^+} V_{D^+}, \tag{12}$$

where $\xi_\Gamma = \mathrm{Tr}(\Gamma) V$.

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Thus, the value of the potential $P_{D^+\Gamma}\xi_\Gamma$ is determined completely by its density ξ_Γ or the clear trace [15].

Next, we obtain the clear trace for the operator L. For this purpose, let us introduce a trace operation as follows. Let Γ_ϵ^+ be smooth manifolds parallel to Γ in the sense of [5, Ch. 2]: $\Gamma_\epsilon^+ \subset D^+$, $\Gamma_\epsilon^+ \to \Gamma$ if $\epsilon \to 0$. Then, the trace operator $\mathrm{Tr}_\Gamma^+ : H^s(D^+) \to H^{s-1/2}(\Gamma)$ is given by

$$\mathrm{Tr}_\Gamma^+ U_{D^+} \stackrel{def}{=} \lim_{\epsilon\to 0} \mathrm{Tr}_{\Gamma_\epsilon^+} U_{D^+}, \tag{13}$$

where

$$\mathrm{Tr}_{\Gamma_\epsilon^+} U_{D^+} \stackrel{def}{=} U_{D^+}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_\epsilon^+.$$

One can also introduce the operator $\mathrm{Tr}_\Gamma^- : H_0^s(D^-) \to H^{s-1/2}(\Gamma)$ in a similar way.

Then, the potential $P_{D^+\Gamma}\xi_\Gamma$ can be determined by the following propositions.

Proposition 1. *If $L_y := L_f$, then*

$$P_{D^+\Gamma}\xi_\Gamma = -L_{D^+}^{-1} A_n \xi_\Gamma \delta(\Gamma), \tag{14}$$

where $\xi_\Gamma \stackrel{def}{=} \mathrm{Tr}(\Gamma)V = \mathrm{Tr}_\Gamma^+ V_{D^+}$, $V \in \Xi_D$, $A_n = \sum A^i n_i$, $\mathbf{n}$ is the outward normal to the boundary Γ.

Proof. For any test function $\Phi \in C_0^\infty(K_T)$, we have

$$\begin{aligned}
\langle LV, \Phi\rangle &= -\sum_1^m \langle A^i V, \nabla_i \Phi\rangle - \langle \nabla A V, \Phi\rangle + \langle BV, \Phi\rangle \\
&= -\sum_1^m \int_T \int_D \left(A^i V, \nabla_i \Phi\right) d\mathbf{x}\, dt - \langle \nabla A V, \Phi\rangle + \langle BV, \Phi\rangle \\
&= -\sum_1^m \left[\int_T \int_{D^-} \left(A^i V, \nabla_i \Phi\right) d\mathbf{x}\, dt + \int_T \int_{D^+} \left(A^i V, \nabla_i \Phi\right) d\mathbf{x}\, dt\right] - \langle \nabla A V, \Phi\rangle + \langle BV, \Phi\rangle \\
&= \left\langle \{LV\}, \Phi\right\rangle + \int_T \int_\Gamma \left(A_n [V]_\Gamma, \Phi\right) d\mathbf{x}\, dt = \left\langle \{LV\}, \Phi\right\rangle + \left\langle A_n [V]_\Gamma \delta(\Gamma), \Phi\right\rangle \\
&= \left\langle L\left(\theta(D^-)V\right), \Phi\right\rangle + \left\langle \theta(D^+) L(V), \Phi\right\rangle - \left\langle A_n \xi_\Gamma \delta(\Gamma), \Phi\right\rangle.
\end{aligned}$$

Here, $\{LV\}$ is the part of LV with the support $D \setminus \Gamma$, $\nabla A \stackrel{def}{=} \sum_1^m \nabla_i A^i$, $[\cdot]_\Gamma$ means the discontinuity across the boundary Γ:

$$[W]_\Gamma \stackrel{def}{=} \mathrm{Tr}_\Gamma^- W - \mathrm{Tr}_\Gamma^+ W.$$

Thus, from Definition 2 we have

$$P_{D^+} V_{D^+} = L_{D^+}^{-1}\left[L\left(\theta(D^-)V\right) - A_n \xi_\Gamma \delta(\Gamma)\right]. \tag{15}$$

Meanwhile, the value of $P_{D^+} V_{D^+}$ does not depend on $\theta(D^-)V$. To prove it, let us consider $\widetilde{V}$: $\widetilde{V}_{D^+} = V_{D^+}$ and $\widetilde{V} \in \Xi_D$. Then, from (15), we obtain $P_{D^+} V_{D^+} - P_{D^+} \widetilde{V}_{D^+} = 0_{D^+}$.

Hence, $\xi_\Gamma = U_\Gamma$ is the clear trace. Thus, we can extend V to D^- by zero and obtain that $P_{D^+\Gamma}\xi_\Gamma = -L_{D^+}^{-1} A_n \xi_\Gamma \delta(\Gamma)$. □

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In order to consider the second-order operator L_s, let us note [24] that

$$\nabla U = \{\nabla U\} + [U]_\Gamma \mathbf{n}\delta(\Gamma).$$

Green's formula for distributions [22,24] reads as by:

$$\Delta U = \{\Delta U\} + \left[\frac{\partial U}{\partial \mathbf{n}}\right]_\Gamma \delta(\Gamma) + \frac{\partial}{\partial \mathbf{n}}\big([U]_\Gamma \delta(\Gamma)\big).$$

Then, one can obtain that

$$\nabla(p\nabla U) = \big\{\nabla(p\nabla U)\big\} + p_\Gamma \left[\frac{\partial U}{\partial \mathbf{n}}\right]_\Gamma \delta(\Gamma) + \frac{\partial}{\partial \mathbf{n}}\big(p_\Gamma [U]_\Gamma \delta(\Gamma)\big),$$

where $p_\Gamma = p(\Gamma)$.

Hence, we have

$$L_s U = \{L_s U\} + p_\Gamma \left[\frac{\partial U}{\partial \mathbf{n}}\right]_\Gamma \delta(\Gamma) + \frac{\partial}{\partial \mathbf{n}}\big(p_\Gamma [U]_\Gamma \delta(\Gamma)\big)$$

and

$$LU = \{LU\} - p_\Gamma \left[\frac{\partial U}{\partial \mathbf{n}}\right]_\Gamma \delta(\Gamma) - \frac{\partial}{\partial \mathbf{n}}\big(p_\Gamma [U]_\Gamma \delta(\Gamma)\big). \tag{16}$$

Thus, it appears that for the second-order operator L_s the potential $P_{D^+\xi_\Gamma}$ is represented by a linear combination of single- and double-layer potentials.

Proposition 2. *If $L_y := L_s$, then*

$$P_{D^+\Gamma}\xi_\Gamma = L_{D^+}^{-1}\left(\big(p_\Gamma \xi_\Gamma^{(2)}\big)\delta(\Gamma) + \frac{\partial}{\partial \mathbf{n}}\big(p_\Gamma \xi_\Gamma^{(1)} \delta(\Gamma)\big)\right), \tag{17}$$

where $\xi_\Gamma = (\xi_\Gamma^{(1)}, \xi_\Gamma^{(2)})^T$ and $\xi_\Gamma \stackrel{def}{=} \mathrm{Tr}(\Gamma)V = \mathrm{Tr}_\Gamma^+(V_{D^+}, \frac{\partial}{\partial \mathbf{n}} V_{D^+})^T$, $V \in \Xi_D$.

The proof of the proposition is similar to the proof of Proposition 1 if we take into account equality (16).

Thus, we arrive at the general result formulated in the next proposition.

Proposition 3. *If $\xi_\Gamma \stackrel{def}{=} \mathrm{Tr}(\Gamma)V$, $V \in \Xi_D$, then the potential $P_{D\Gamma}\xi_\Gamma$ is determined and*

$$P_{D\Gamma}\xi_\Gamma = -L_{D^+}^{-1}\big(A_\Gamma \zeta(\xi_\Gamma)\big). \tag{18}$$

If $L_y := L_f$, then $\xi_\Gamma = V_\Gamma$, $\zeta(\xi_\Gamma) = V_\Gamma \delta(\Gamma)$ and $A_\Gamma = A_n$.

If $L_y := L_s$, then $\xi_\Gamma = (V_\Gamma, \frac{\partial}{\partial \mathbf{n}} V)^T$, $\zeta(\xi_\Gamma) = (V_\Gamma \frac{\partial}{\partial \mathbf{n}} \delta(\Gamma), \frac{\partial}{\partial \mathbf{n}} V \delta(\Gamma))^T$ and $A_\Gamma = -p_\Gamma(1, 1)$.

Thus, in Proposition 3, the clear trace $\xi_\Gamma \in \Xi(\Gamma)$ is represented by the Cauchy data. As we see, the clear trace is only determined by the spatial-differential part of the operator L.

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Next, we obtain an important property of the potential used in further consideration. Along with "friendly" field U^+, let us consider "adverse" field $U^- : U^- = L^{-1}\bar{f}^-$. Then, one can immediately verify the following important properties of the potential:

$$P_{D^+}U^+_{D^+} = 0_{D^+}, \tag{19}$$

and

$$P_{D^+}U^-_{D^+} = U^-_{D^+}. \tag{20}$$

These properties were first obtained in the stationary formulation in [15,18].

Hence, from the linearity of the problem we obtain that:

$$P_{D^+}U_{D^+} = U^-_{D^+}. \tag{21}$$

Thus, the field generated in D^+ does not contribute to the potential, while the field generated outside D^+ is projected by the operator P_{D^+} onto itself. Hence, the operator P_{D^+} is a projection.

4. General solution of AC problem

On the basis of the potential introduced in the previous section one can obtain the general solution of the AC problem formulated in Section 2.

From (19) we have that the requirements of the noise cancelation is equivalent to the following equality:

$$P_{D^+}L^{-1}_{D^+}(f + G) = P_{D^+}L^{-1}_{D^+}f^+ = 0_{D^+}. \tag{22}$$

This implies that the total field generated by both f and G on D^+ is equivalent to the field only generated on D^+. Meanwhile, the field generated on D^+ should not contribute to the potential.

Next, from Proposition 3, it follows that

$$P_{D^+}L^{-1}_{D^+}f = P_{D^+\Gamma}\xi_\Gamma = L^{-1}_{D^+}\big(-A_\Gamma\zeta(\xi_\Gamma)\big), \tag{23}$$

where $\xi_\Gamma = \mathrm{Tr}(\Gamma)U$.

On the other hand, from (20) we have:

$$P_{D^+}L^{-1}_{D^+}G = L^{-1}_{D^+}G. \tag{24}$$

Thus, from (22), (23) and (24), we obtain:

$$L^{-1}_{D^+}\big(G - A_\Gamma\zeta(\xi_\Gamma)\big) = 0_{D^+}. \tag{25}$$

The general solution of (25) is given by

$$G = A_\Gamma\zeta(\xi_\Gamma) + G_v, \tag{26}$$

where $G_v = LV_a$ and V_a is an arbitrary function: $V_a \in \Xi_D$, $\operatorname{supp} V_a \subset \overline{D^-}$.

In (26), the first term corresponds to a source distributed on the boundary Γ, while the second term includes volume source terms.

In particular, it is possible to retain only the surface potential source:

$$G_0 = A_\Gamma\zeta(\xi_\Gamma). \tag{27}$$

By immediate substitution, one can prove that the solution of BVP (4) is the following:

$$W(\mathbf{x},t)=\begin{cases}U^{+}(\mathbf{x},t), & \mathbf{x}\in D^{+},\\ U(\mathbf{x},t)+U^{+}(\mathbf{x},t), & \mathbf{x}\in D^{-}.\end{cases}$$

One can show that the source term (27) gives us, in particular, the AC sources obtained in [9] and [17] be means of the theory of distributions.

It is to be noted that the simple alteration of the normal sign immediately gives us the insulation of the domain D^- from the field generated in D^+.

In the next two sections we consider several examples of AC sources for stationary and non-stationary problems. We assume that some boundary conditions are set for each problem and the appropriate BVP is well-posed. The AC source term, representing the general solution (27), is obtained for the Maxwell equations and aeroacoustic equations.

5. Maxwell equations. Electro- and magnetostatics problems

In this section we obtain the AC sources for the electrostatics and magnetostatics equations. For this purpose, the classical problems on a bounded conductor in an electrostatic field and bounded superconductor in a magnetostatic field are interpreted as AC problems.

5.1. A bounded conductor in electrostatic field

Consider a bounded conductor in an external electrostatic field $\mathbf{E}_{out}$. Since there is no current inside the body (the problem is static), the internal electric field must be equal to zero. The zero state of this field is reached via the redistribution of the charges on the body surface.

Let us consider this problem as an AC problem. Then, the contribution of the surface charges is similar to shielding the body from the external field $\mathbf{E}_{out}$. From the Maxwell equations for an electrostatic field it follows that

$$div\,\mathbf{E}=4\pi\rho+g_d, \tag{28}$$

$$curl\,\mathbf{E}=\mathbf{g}_c. \tag{29}$$

Here, $\mathbf{E}$ is the electric field, ρ is the density of charges, g_d and $\mathbf{g}_c$ are the AC source terms.

Consider Eqs. (28), (29) as a partial case of the governing equation in (4). Then, $f^-=4\pi\rho$, and the appropriate matrix A_n from Proposition 1 is given by

$$A_n=\begin{pmatrix} n_1 & n_2 & n_3\\ 0 & -n_3 & n_2\\ n_3 & 0 & -n_1\\ -n_2 & n_1 & 0\end{pmatrix}, \tag{30}$$

if $U=(E_1,E_2,E_3)^T$, where $E_i\ (i=1,2,3)$ are the coordinates of the vector $\mathbf{E}$ in some Cartesian coordinate system.

From Proposition 1 we obtain that

$$g_d=\mathbf{E}_{out|\Gamma}\cdot\mathbf{n}\delta(\Gamma),$$

$$\mathbf{g}_c=\mathbf{n}\times\mathbf{E}_{out|\Gamma}\delta(\Gamma),$$

where $\mathbf{E}_{out|\Gamma}$ is the external field on the perimeter of the body.

The external field $\mathbf{E}_{out|\Gamma}$ must be orthogonal to the boundary, otherwise there is current on the surface. Hence, $\mathbf{g}_c\equiv 0$.

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Let us now represent the AC source g_d in the following form: $g_d = 4\pi\sigma_\rho\delta(\Gamma)$. Then, the distribution of the surface charges is given by

$$\sigma_\rho = \frac{1}{4\pi} E_{out|\Gamma}. \tag{31}$$

5.2. A bounded superconductor in magnetostatic field

It is well known that the magnetic field does not penetrate inside a superconductor. Hence, it must be tangential to the boundary immediately outside the superconductor [4].

If we interpret this problem as an AC problem, then, from the Maxwell equations for a static magnetic field, we have:

$$div\,\mathbf{H} = g_d, \tag{32}$$

$$curl\,\mathbf{H} = \frac{4\pi}{c}\sigma\mathbf{E} + \mathbf{g}_c. \tag{33}$$

Here, $\mathbf{H}$ is the magnetic field, σ is the conductivity, c is the speed of light, g_d and $\mathbf{g}_c$ are the AC source terms.

It is clear that the operator L and appropriate matrix A_n are the same as in the previous example. Hence, we obtain that

$$g_d = H_{n|\Gamma}\delta(\Gamma),$$

$$\mathbf{g}_c = \mathbf{n} \times \mathbf{H}_\Gamma\delta(\Gamma),$$

where $H_n = \mathbf{H} \cdot \mathbf{n}$.

Since the magnetic field is tangential to the boundary, we have $g_d \equiv 0$.

Let us set that $\mathbf{g}_c = \frac{4\pi}{c}\mathbf{j}_b\delta(\Gamma)$. Then, from Eq. (32) we obtain the bound current density:

$$\mathbf{j}_b = \frac{c}{4\pi}\mathbf{n} \times \mathbf{H}_\Gamma. \tag{34}$$

Thus, the magnetic external field induces a bound current $\mathbf{j}_b$ which can be interpreted as an AC source.

The AC problem, closely related to the active noise shielding problem, is addressed in the next section.

6. Aeroacoustics. Active noise shielding

Let us first consider acoustic equations represented by the wave equation.

6.1. Wave equation

In the case of the wave equation

$$u_{tt} - \nabla(p\nabla u) - qu = f, \tag{35}$$

the operator L in (1) is the following: $L := \frac{\partial^2}{\partial t^2} + L_s$.

From the general solution (27) and Proposition 3, we immediately obtain the AC source term:

$$g_0 = -p_\Gamma \frac{\partial u}{\partial \mathbf{n}}_{|\Gamma}\delta(\Gamma) - \frac{\partial p_\Gamma u_\Gamma \delta(\Gamma)}{\partial \mathbf{n}}. \tag{36}$$

The particular case of a time-harmonic wave $u = ve^{i\omega t}$ results in the source for the Helmholtz equation obtained in [22].

The general case of aeroacoustics is described by the LEE.

6.2. Linearized Euler equations

In the case of the LEE model, we consider small acoustic perturbations of the mean flow. Then, neglecting both viscous terms and high-order terms with respect to the perturbations, we arrive at the following set of equations:

$$\frac{1}{\rho_0 c_0^2}\left(p'_t + (\mathbf{u}_0, \nabla)p'\right) + \frac{1}{\rho_0 c_0^2}(\mathbf{u}', \nabla)p_0 + \nabla \cdot \mathbf{u}' = \frac{1}{\rho_0 c_0^2} f^{(p)} + q_{vol},$$
$$\rho_0\left(\mathbf{u}'_t + (\mathbf{u}_0, \nabla)\mathbf{u}' + (\mathbf{u}', \nabla)\mathbf{u}_0\right) + \nabla p' = \mathbf{f}^{(u)} + \mathbf{f}_{vol}, \tag{37}$$

where u'_j $(j = 1, 2, 3)$ are the components of the particle velocity $\mathbf{u}'$ in some Cartesian coordinate system; p' is the sound pressure; c_0 is the speed of sound; the functions marked by 0 correspond to some main flow; q_{vol} is the volume velocity per a unit volume and $\mathbf{f}_{vol}$ is the force per a unit volume [12]; $f^{(p)}$ and $\mathbf{f}^{(u)}$ are possible additional sound sources.

If we set

$$U = \left(u'_1, u'_2, u'_3, p'\right)^T, \tag{38}$$

then the matrix A_n is given by

$$A_n = \begin{pmatrix} n_1 & n_2 & n_3 & \frac{u_n}{\rho_0 c_0^2} \\ \rho_0 u_n & 0 & 0 & n_1 \\ 0 & \rho_0 u_n & 0 & n_2 \\ 0 & 0 & \rho_0 u_n & n_3 \end{pmatrix}, \tag{39}$$

where $u_n = \mathbf{u}_0 \cdot \mathbf{n}$.

Thus, we obtain the following AC sources:

$$q_{vol} = \left(\mathbf{u}' \cdot \mathbf{n}_{|\Gamma} + \frac{u_n}{\rho_0 c_0^2} p'_{|\Gamma}\right)\delta(\Gamma),$$
$$\mathbf{f}_{vol} = \left(p'_{|\Gamma}\mathbf{n} + \rho_0 u_n \mathbf{u}'_{|\Gamma}\right)\delta(\Gamma). \tag{40}$$

It can be seen that the flux through the boundary Γ affects the AC sources (40).

If the main flow is at rest (acoustic equations), then AC sources (40) coincide with the active noise shielding source terms obtained in [17]. Eliminating the particle velocity $\mathbf{u}'$ in the acoustic equations, one can obtain the source (36) from both (37) and (40).

7. Conclusion

The potentials introduced by Ryaben'kii have been generalized to IBVPs and the theory of distributions. The clear traces, assigned with the potentials, have been found for the first- and second-order spatial differential operators. The general solution of a nonstationary AC problem has been obtained using the theory of the potentials. In particular, this solution is applicable to the problems of electromagnetics and aeroacoustics. The general solution of the AC problem has been applied to the classical problems on a bounded conductor in electrostatic field and a bounded superconductor in a magnetostatic field. In aeroacoustics, the AC problem is reduced to active noise shielding. The control sources have been obtained in the form of single- and double-layer sources. The solution only requires the knowledge of the total field ("friendly" and noise) on the perimeter of the shielded domain.

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