IMA Journal of Numerical Analysis (2005) Page 1 of 15
doi: 10.1093/imanum/dri017

Active screening model for hyperbolic equations

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[Received on 27 December 2005]

The problem of active screening in application to hyperbolic equations is analysed. According to the problem, two domains effecting each other via distributed sources terms are considered. It is required to implement additional sources nearby the common boundary of the domains to "isolate" one domain from the other one despite of all the original sources are remained. It is important that the location of the original sources is unknown whereas the field of their action nearby the boundary is only known.
In the paper, the theory of difference potentials is applied to the system of hyperbolic equations. It allows one to obtain a single–layer active screening not requiring any additional computations. Local single–layer and double–layer active screening sources are obtained for an arbitrary hyperbolic system. The solution does not require either the knowledge of the Green's function or specific characteristics of the sources and medium. The optimal single–layer active screening solution is derived in the case of a free space. In particular, the results are applicable to the system of acoustics equations. The questions related to a practical realization including the mutual location of the primary and secondary sources, as well as the measurement point, are discussed. The active noise shielding can be realized via a single–layer source term requiring the measurements only at one layer nearby the domain screened.

Keywords: Active screening, noise shielding, method of difference potentials, hyperbolic equations.

1. Introduction

The problem of active screening (AS) for hyperbolic equations is analysed. According to the problem, two domains effecting each other via distributed sources terms are considered. It is required to distribute additional sources nearby the common boundary of the domains to "isolate" one domain from the other despite of all the original sources are remained. It is important that the location of the original sources is unknown whereas the field of their action nearby the boundary is only known.

This problem is closely related to the problems of active noise and vibration reduction. In the framework of the active noise shielding, the implementation of additional acoustic sources is conducted in such a way that the total acoustic undesirable noise in the protected domain is decreased. In contrast to "passive" noise reduction, the "active" shielding does not assume any mechanical disconnection of the domain shielded, therefore it may be potentially more flexible. Meanwhile, both these approaches can be successfully combined because mechanical obstacles are efficient in shielding from high frequencies while the active shielding approach is easier for realization in the case of low frequencies. In turn, in the active vibration control additional vibration sources are distributed along the perimeter of the domain where the total vibration is to be decreased (or eliminated).

Both the active noise shielding and active vibration control a relatively new directions for research. First theoretical papers on active shielding were published only about 30 years ago (see Jessel (1968), Malyuzhinets (1992), Fedoryuk (1976)), whereas first publications on some possible implementations appeared much later (see Burgess (1981), Elliot *et al.* (1987)). Most of active shielding techniques are based on sound control in selected discrete (see Burgess (1981), Elliot *et al.* (1987), Cabell & Fuller (1998)) or directional areas (see Wright & Vuksanovic (1997)). Many approaches, e.g. Kincaid & Laba (1998), assume detailed information about the sources of noise. There is a number of publications devoted to the optimization of the strengths of spatially distributed secondary sources (see Nelson *et al.* (1987)). Comprehensive reviews of theoretical and practical approaches to both active noise shielding and active vibration control can be found in books (see Nelson & Elliott (1992), Fuller *et al.* (1996), Tochi & Veres (2002)). One of major substantial drawbacks of the standard approaches is the requirement of information about characteristics of ”adverse” sources including their location. It is worth noting that this information often is not available in practice.

In acoustics, the problem becomes much more complicated if some ”friendly” field (sound) is assumed to be in the protected area. In this case, along with shielding from the ”adverse” field, the factorisation of ”friendly” and ”adverse” components is to be performed. The suppression of the ”adverse” field may be not sufficient if it substantially damages the ”friendly” component in the shielded domain. There is a separate class of methods that requires the information on the total field (both ”friendly” and ”adverse”) only at the perimeter of the domain to be shielded if the problem is linear. It is to be noted that the knowledge of both the ”adverse” and ”friendly” components is not required. These approaches are mostly based on the knowledge of Green' function. For instance, the exact solution of the active shielding problem is obtained by Malyuzhinets (1992) for the Helmholtz equation with constant coefficients. The general surface–potential solution of the AS problem for a linear analogue of the Helmholtz equation with variable coefficients was obtained by Tsynkov (2003). The solution is formulated as a superposition of the surface single–layer and double–layer potentials; in the general formulation it requires the knowledge of the perimeter distribution of both the field function and its normal derivative. The approach founded on the Difference Potential theory (see Ryaben'kii (1995), Ryaben'kii (2002)) provides a general approach to solving the AS problem in a finite–difference formulation. This solution is applicable to arbitrary geometric configurations, medium and boundary conditions. In contrast to the other methods described above, the ultimate AS solution is achieved in a finite–difference form. From a practical standpoint this may not be necessarily treated as a drawback because the implementation of the active shielding assumes some discrete distribution of active shielding sources. This approach has been analysed in application to the Helmholtz equation in (see Loncaric *et al.* (2001), Loncaric *et al.* (2003), Loncaric *et al.* (2004), Loncaric *et al.* (2005)). In these papers the active shielding is mostly obtained in a double–layer form that assumes numerical second order differentiation of measurement values. On the basis of the development of finite–difference surface potentials for the Helmholtz equation (or its analogue), in Tsynkov (2003) it is suggested a single–layer active shielding but its realization requires solving some external problem.

In the current paper, the theory of difference potentials is applied to the system of hyperbolic equations. In particular, the obtained results are applicable to the system of acoustics equations. It allows us to obtain a single–layer active shielding which does not require any additional computations. Instead, it may require the measurement of an additional physical value that is the normal component of the velocity. It is to be noted that the active shielding source terms do not include subtraction of measurement data. The questions related to a practical realization including the location of the secondary sources, as well as the measurement point, are discussed. Local single–layer and double–layer AS sources are obtained for an arbitrary hyperbolic system. The optimal single–layer AS solution is derived in the case of a free space.

2. Statement of the active screening problem

Mathematical formulation of the AS problem can be done in the following form. Let us assume that field (sound) propagation is described by some linear boundary value problem in a domain D_0:

$$Lw = S, \tag{2.1}$$

$$w \in U_{D_0}, \tag{2.2}$$

where U_{D_0} is a linear subspace of functions defined on $\overline{D}_0$ such that the solution of problem (2.1), (2.2) exists and unique.

In particular, the domain D_0 may be a free space, and the boundary condition (2.2) can be represented by the Sommerfeld boundary conditions (see Loncaric *et al.* (2001)).

Let us consider now some domain D such that $D \subset D_0$. The sources on the right-hand side can be located both in D and outside of D:

$$\begin{aligned} S &= S_f + S_a, \\ \sup S_f &\subset D, \\ \sup S_a &\subset D_0 \diagdown D. \end{aligned} \tag{2.3}$$

Here, S_f is supposed to be the source of desirable (”friendly”) field, then S_a is the source of undesirable (”adverse”) field.

Such a partition of the field on ”friendly” and ”adverse” components is given for the sake of determinicity. It is possible also to consider a somewhat opposite formulation where the domain $D_0 \backslash D$ is shielded from the field coming from the domain D.

Suppose that we know the distribution of the function w in some vicinity of the boundary of D. It is important to emphasize that only this information is assumed to be available. In particular, the distribution of the sources S on the right–hand side is unknown. The AS problem is reduced to seeking additional sources g in $D_0 \diagdown D$ such that the solution of problem

$$Lw = S + g, \tag{2.4}$$

$$\sup g \in D_0 \backslash D,$$

$$w \in U_{D_0} \tag{2.5}$$

coincides with the solution of problem (2.1), (2.2) on the subdomain D if $S = S_f$. It is worth noting that an ”obvious” solution $g = -S_a$ is not appropriate here because the distribution of S_a is unknown. Moreover, if the density S_a is known, the trivial solution $g = -S_a$ seems not to be realistic for a practical realization.

3. Difference potential formalism and main theorem

Following the difference potential method by Ryaben’kii (2002), let us consider some grid M^0 in D_0. Next, introduce subsets of grid M^0 as follows: $M^+ = M^0 \cap \overline{D}$, $M^- = M^0 \backslash M^+$. Assume that equation (2.1) is approximated on some stencil N_m by equation

$$L_h w^{(h)}|_m \equiv \sum_n a_{mn} w_n^{(h)} = S_{|m}^{(h)}, \;\; m \in M^0, \;\; n \in N_m. \tag{3.1}$$

Equation (3.1) is completed by the boundary condition approximating the continuous boundary condition (2.2):

$$w^{(h)} \in U^{(h)}_{D_0}, \tag{3.2}$$

where $U^{(h)}_{D_0}$ is a linear discrete space of functions defined on M^0 that is a discrete counterpart of the space U_{D_0}.

Denote the extensions of the sets M^0, M^+, M^- due to the stencil by N^0, N^+, N^-, respectively. It is clear that the sets N^+ and N^- intersect each other. We treat their intersection as the grid boundary γ of the domain D: $\gamma = N^+ \bigcap N^-$. In turn, the grid boundary γ is split into two nonintersecting subboundaries γ^- and γ^+: $\gamma = \gamma^+ \cup \gamma^-$, where $\gamma^+ = \gamma \bigcap \overline{D}$ and $\gamma^- = \gamma \backslash \gamma^+$. Now the finite–difference solution of the finite–difference counterpart of the AS problem (2.4), (2.5) is to be founded.

The AS problem is then formulated in a finite–difference form as follows. We consider problem (3.1), (3.2) where

$$\begin{aligned} &S^{(h)}_{|m} = S^{(h)}_{f|m} + S^{(h)}_{a|m}, \\ &\sup S^{(h)}_f \in M^+, \\ &\sup S^{(h)}_a \in M^-. \end{aligned} \tag{3.3}$$

It is required to find such an additional source term

$$g^{(h)} : \sup g^{(h)} \in M^- \tag{3.4}$$

that the solution of problem

$$\begin{aligned} &L_h w^{(h)}|_m \equiv \sum_n a_{mn} w^{(h)}_n = S^{(h)}_{|m} + g^{(h)}, \;\; m \in M^0, \\ &w^{(h)} \in U^{(h)}_{D_0} \end{aligned} \tag{3.5}$$

coincides on N^+ with the solution of problem (3.1), (3.2) if $S^{(h)}_a \equiv 0$. The function w_γ is assumed to be known.

The general solution of the AS problem in the discrete formulation is provided by the following main theorem by Ryaben'kii (1995):

$$g_h = -L_h v^{(h)}|_{M^-}, \;\; m \in M^- \tag{3.6}$$

$$g_h = 0, \;\; m \in M^+, \tag{3.7}$$

where $v^{(h)}$ is an arbitrary function such that

$$v^{(h)} \in U^{(h)}_{D_0}, \quad v^{(h)}_\gamma = w^{(h)}_\gamma. \tag{3.8}$$

The prove of this theorem can be found in Ryaben'kii (2002). It is clear that the function $v^{(h)}$ in (3.6) is not unique. A partial case of this function corresponds to $v^{(h)}|_{M^0 \backslash \gamma} = 0$. In this case the AS source term is only located on the minimal possible support consisting of such nodes that $\gamma \bigcap N_m \neq \emptyset$, $m \in M^-$. It is important to note here that in contrast to the continuous case the grid boundary is not necessarily to be a single–layer.

4. 1D acoustics system

Let us first consider the example of 1D acoustics system for isentropic flows:

$$p_t + \rho c^2 u_x = f^{(p)}, \quad (4.1)$$
$$u_t + p_x/\rho = f^{(u)}.$$

Here, p is the pressure, ρ is the density of air, u is the velocity and c is the sound speed, $f^{(p)}$ and $f^{(u)}$ are acoustic sources.

Assume that the acoustic sources are time-harmonic:

$$f^{(p)} = \rho c^2 \widehat{f}^{(p)} e^{i\omega t}, \quad (4.2)$$
$$f^{(u)} = c \widehat{f}^{(u)} e^{i\omega t}.$$

Hence, the dependent variables can be represented in the Fourier harmonics:

$$p = \rho c \widehat{p} e^{i\omega t}, \quad (4.3)$$
$$u = \widehat{u} e^{i\omega t}.$$

Then, the equations for the Fourier images are as follows:

$$ik\widehat{p} + \widehat{u}_x = \widehat{f}^{(p)}, \quad (4.4)$$
$$ik\widehat{u} + \widehat{p}_x = \widehat{f}^{(u)}.$$

Let us consider AS problem in a free space with the following Sommerfeld boundary condition:

$$\left(ikI + C\frac{\partial}{\partial x}\right)\widehat{W}_{|\infty} = 0, \quad (4.5)$$

where

$$\widehat{W} = (\widehat{u},\ \widehat{p})^T, \quad C = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

This boundary condition means that the Riemann invariant $\widehat{R}^+$ is remained whereas $\widehat{R}^- = 0$.

The system for the Fourier images can be easily written in the characteristic form:

$$L^+\widehat{R}^+ = \widehat{f}^+, \quad (4.6)$$
$$L^-\widehat{R}^- = \widehat{f}^-,$$

where $L^+ = ik + \frac{\partial}{\partial x}$, $L^- = ik - \frac{\partial}{\partial x}$, $\widehat{R}^+ = \widehat{p} + \widehat{u}$, $\widehat{R}^- = \widehat{p} - \widehat{u}$, $\widehat{f}^+ = \widehat{f}^{(p)} + \widehat{f}^{(u)}$, $\widehat{f}^- = \widehat{f}^{(p)} - \widehat{f}^{(u)}$. The functions $\widehat{R}^+$ and $\widehat{R}^-$ are the Fourier images of the Riemann invariants of system (4.1) propagating along the characteristics $\frac{dx}{dt} = c$ and $\frac{dx}{dt} = -c$, accordingly.

We approximate these equations with account of the hyperbolic properties of the original equations written in the characteristics. It can be done if we consider the following ”upwind” approximation:

$$\left(ik + \frac{\nabla}{h}\right)\widehat{R}^+_m = \widehat{f}^+_m, \quad (4.7)$$

$$\left(ik - \frac{\Delta}{h}\right)\widehat{R}_m^- = \widehat{f}_m^-, \tag{4.8}$$

where $\nabla s_m = s_m - s_{m-1}$ and $\Delta s_m = s_{m+1} - s_m$.

We assume that the area $x < 0$ is shielded and in the discrete space it corresponds to $m < 0$. Then, in the case of equation (4.7) for $\widehat{R}^+$ the boundary $\gamma \equiv \gamma_+$ is one-layer and corresponds to $m = -1$. In turn, in the case of equation (4.8) the boundary $\gamma \equiv \gamma^-$ is also one-layer and coincides with the point $m = 0$.

Equation (4.7) for $\widehat{R}^+$ does not require any source terms to provide the active shielding. It follows from the physical meaning of this equation because it only describes the transfer of information (sound) from the area shielded. This result is formally derived below. It is important to note that this conclusion is based on the free–space statement of the problem. If the domain D_0 is bounded, the active shielding source term in the $\widehat{R}^+$– Riemann invariant equation appears to be responsible for retaining the "friendly" field reflected from the external boundary of D_0 ("echo"– effect) and might be substantial.

According to the main theorem, the AS source $\widehat{g}^{(h)}$ is given by:

$$\widehat{g}^{(h)} = -L_h V^{(h)}_{|M^-}, \tag{4.9}$$

$$V^{(h)}_\gamma = (\widehat{R}^+, \widehat{R}^-)^T_\gamma, \tag{4.10}$$

$$\left(ikI + \frac{1}{h}\widehat{C}\nabla\right)V^{(h)}_{|\infty} = 0, \tag{4.11}$$

$$\widehat{C} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The vector $V^{(h)} \equiv (v_1^{(h)}, v_2^{(h)})^T$ satisfying to (4.9) is non-unique. It is possible to choose the vector $V^{(h)}$ with the following components:

$$v^{(h)}_{1|\gamma^+} = \widehat{R}^+_{-1}, v^{(h)}_{2|\gamma^-} = \widehat{R}^-_0, \tag{4.12}$$

$$(ikh + \nabla)v^{(h)}_{1|M^-} = 0, v^{(h)}_{2|N^-\setminus\gamma} = 0.$$

It is easy to see that vector (4.12) satisfies boundary conditions (4.10), (4.11). Though the boundary value problem is formulated for the function $v_1^{(h)}$, it is not to be solved since its solution is not explicitly used.

This immediately leads us to the following active shielding single–layer source term formulated at the boundary γ^-:

$$\widehat{g}^{(h)}_{u|0} \equiv (\widehat{g}_0^+,\ \widehat{g}_0^-)^T = \left(0,\ -(ik + \frac{1}{h})\widehat{R}_0^-\right)^T. \tag{4.13}$$

This result also has a clear physical interpretation since the field is distributed along the characteristics incoming to the domain shielded. This information is fully contained in the $\widehat{R}^-$– Riemann invariant. In practice, we can effectively explore the fact that the Riemann invariant is remained along a characteristic. On the basis of this property it is possible to measure the value of the Riemann invariant in the vicinity of the boundary at any place where it is convenient to perform it. In fact, it demands the measurement of the both dependent variables which are the pressure and velocity since the Riemann invariants are not the physical values. If "friendly" sound is absent, then $\widehat{R}_0^+ = 0$ and the active shielding requires the measurement of only one physical value, e.g., the pressure:

$$\widehat{g}^{(h)}_{u|0} = \left(0,\ -2(ik + \frac{1}{h})\widehat{p}_0\right)^T. \tag{4.14}$$

It is worth noting that if the space D_0 is bounded, then, instead of (4.12) in (4.9), it is possible to choose the following function $V^{(h)}$:

$$V^{(h)}_{|N^-\setminus\gamma} = 0 \tag{4.15}$$

Then, the active shielding source term becomes as follows:

$$\widehat{g}^{(h)}_{b|0} = \left(\frac{1}{h}\widehat{R}^+_{-1},\ -(ik+\frac{1}{h})\widehat{R}^-_0\right)^T. \tag{4.16}$$

The upwind approximation (4.7), (4.8) can be easily written in the original variables $\widehat{u}$ and $\widehat{p}$ after adding and subtracting equations (4.7), (4.8):

$$ik\widehat{p}_m + \frac{\dot{\Delta}}{2h}\widehat{u}_m = \widehat{f}^{(p)}_m + \frac{\Delta_2}{2h}\widehat{p}_m, \tag{4.17}$$

$$ik\widehat{u}_m + \frac{\dot{\Delta}}{2h}\widehat{p}_m = \widehat{f}^{(u)}_m + \frac{\Delta_2}{2h}\widehat{u}_m, \tag{4.18}$$

where $\dot{\Delta}s_m = s_{m+1} - s_{m-1}$ and $\Delta_2 s_m = s_{m+1} - 2s_m + s_{m-1}$.

In the original variables, the AS free–space vector is given by:

$$\widehat{G}^{(h)}_{u|0} = \frac{1}{2}(\widehat{g}^-_0, -\widehat{g}^-_0)^T. \tag{4.19}$$

It is clear that this result is somewhat ”optimum” in a free space since it is obtained by the intermediate analysis of spreading the Riemann invariants. Indeed, any vector $\widehat{g}^{(h)}_{b|0}$ is suitable for the active shielding in a free space. At the limit of $kh << 1$, in the original variables this vector equals

$$\widehat{G}^{(h)}_{b|0} = \frac{1}{2h}(\widehat{R}^-_0 - \widehat{R}^+_0, \widehat{R}^-_0 + \widehat{R}^+_0)^T, \tag{4.20}$$

and its first norm is given by

$$\left\|\widehat{G}^{(h)}_{b|0}\right\|_1 = \frac{1}{2h}(|\widehat{R}^-_0 + \widehat{R}^+_0| + |\widehat{R}^-_0 - \widehat{R}^+_0|). \tag{4.21}$$

The minimum of the norm is reached at $\widehat{R}^+_0 = 0$. It immediately follows from an inequality:

$$|\widehat{R}^-_0| \leqslant \frac{1}{2}(|\widehat{R}^-_0 + \widehat{R}^+_0| + |\widehat{R}^-_0 - \widehat{R}^+_0|) \tag{4.22}$$

Thus,

$$\left\|\widehat{G}^{(h)}_{u|0}\right\|_1 \leqslant \left\|\widehat{G}^{(h)}_{b|0}\right\|_1. \tag{4.23}$$

The minimum of the active shielding source term is also reached at $\widehat{R}^+_0 = 0$ in L_2. It is proved below in a more general case.

Let us try now to obtain the same result via the formal procedure. According to general solution (3.6), the vector of the source terms is given by

$$\widehat{G}^{(h)} = -[ikI - \frac{\Delta_2}{2h} + \frac{\dot{\Delta}}{2h}\widehat{A}]V^{(h)}, \tag{4.24}$$

$$\widehat{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with the function $V^{(h)}$ satisfying to boundary conditions (4.10), (4.11).

Since the boundary γ is a double–layer for the entire system (4.17), (4.18), from the formal procedure it is not clear that the active shielding source term is only localized at the point $m=0$ unless the characteristic form is considered. To develop a general approach, the formal analysis of AS is provided for a general hyperbolic 1D system in the next section.

5. AS for 1D hyperbolic system

Let us consider the general linear system of the hyperbolic type and its Fourier image:

$$ik\widehat{W} + A\widehat{W}_x = 0, \tag{5.1}$$

where we assume that

$$A = S\Lambda S^{-1} \tag{5.2}$$

and the matrix $\Lambda = diag\{\lambda_k\}$ is a diagonal matrix of the real eigenvalues of the matrix A.

For the vector of the Riemann invariants $\widehat{Y} = S^{-1}\widehat{W}$, the upwind approximation can be written in the following form:

$$ik\widehat{Y}_m + \frac{1}{h}\Lambda^{+}\nabla\widehat{Y}_m + \frac{1}{h}\Lambda^{-}\Delta\widehat{Y}_m = 0, \tag{5.3}$$

where $\Lambda^{+} = \frac{1}{2}(\Lambda + |\Lambda|), \Lambda^{-} = \frac{1}{2}(\Lambda - |\Lambda|), |\Lambda| = diag\{|\lambda_k|\}$.

It can be rewritten as follows:

$$ik\widehat{Y}_m + \frac{\Lambda}{2h}\dot{\Delta}\widehat{Y}_m = \frac{|\Lambda|}{2h}\Delta_2\widehat{Y}_m, \tag{5.4}$$

here $\dot{\Delta}S_m = S_{m+1} - S_{m-1}, \;\; \Delta_2 S_m = S_{m+1} - 2S_m + S_{m-1}$.

Having turned back to the original variables, we get the "upwind" scheme for the Fourier images:

$$ik\widehat{W}_m + \frac{A}{2h}\dot{\Delta}\widehat{W}_m = \frac{1}{2h}S|\Lambda|S^{-1}\Delta_2\widehat{W}_m. \tag{5.5}$$

Further, we follow the previous section approach. Having obtained the AS source terms for the Riemann invariants, they can be written in the physical variables. Thus, we first split the AS source term as follows:

$$S^{-1}\widehat{G}^{(h)}_{\gamma^-} = \frac{\Lambda}{h}I^{+}_{\lambda}\widehat{Y}_{\gamma^+} + (ikI - \frac{\Lambda}{h})I^{-}_{\lambda}\widehat{Y}_{\gamma^-}, \tag{5.6}$$

where $I^{-}_{\lambda} = diag\{sign\lambda^{-}\}$ is a diagonal matrix having nonzero elements equaled to –1 if the appropriate eigenvalue is negative. We consider two possible values of the matrix I^{+}_{λ}. If the auxiliary function $V^{(h)}$ is defined according to (4.15), then $I^{+}_{\lambda} = diag\{sign\lambda^{+}\}$, similarly to I^{-}_{λ}. In the case of a free space, the optimal solution corresponds to the function $V^{(h)}$ not including the contribution of the outcoming Riemann invariants, hence $I^{+}_{\lambda} \equiv 0$.

In the physical variables the source term is given by:

$$\widehat{G}^{(h)}_{b|\gamma^-} = \frac{1}{h}AB^{+}\widehat{W}_{\gamma^+} + (ikI - \frac{1}{h}A)B^{-}\widehat{W}_{\gamma^-}, \tag{5.7}$$

where the projection matrixes B^{+} and B^{-} are as follows:

$$B^{-} = SI^{-}_{\lambda}S^{-1}, B^{+} = SI^{+}_{\lambda}S^{-1}.$$

Whereas, in a free space the AS solution is the following:

$$\widehat{G}^{(h)}_{u|\gamma^-} = (ikI - \frac{A}{h})B^-\widehat{W}_{\gamma^-}. \tag{5.8}$$

Thus, in 1D case the AS source term can be locally approximated only at γ^-. Formula (5.8) automatically takes into account only incoming invariants. Only derivatives (their approximations) along the incoming characteristics are included in (5.8) . All incoming Riemann invariants are remained along the appropriate incoming characteristics unless source terms exist. Therefore, they can be measured at any point between the boundary, where the AS is implemented, and the nearest source. Though the entire vector W is to be measured, it can be represented as a superposition of both incoming and outcoming Riemann invariants. The latter Riemann invariants do not effect on the AS source terms and their contribution is automatically filtered by formula (5.8).

In contrast to the Helmholtz equation analysis, the AS is reached on a single–layer without solving any external problem. It is worth noting that in the case of the second-order-equation analysis the AS having a compact support includes a subtraction of two measurable values divided by h^2.

Let us assume that we measure the acoustic field at the layer γ^-_+ immediately outside of the boundary γ^- and the domain D. In the 1D example it corresponds to the point $m = 1$. The vectors at γ^- and γ^-_+ can be related each other via the Δ-approximation:

$$(ikh + \Lambda\Delta)\widehat{Y}_{\gamma} = 0$$

or

$$\widehat{Y}_{\gamma^-} = (\Lambda - ikhI)^{-1}\Lambda\widehat{Y}_{\gamma^-_+}. \tag{5.9}$$

This relation does not consistent for the outcoming invariants but they do not effect on our analysis. Then, having substituted (5.9) in (5.8), we are able to formulate the AS term in the original Fourier variables as follows:

$$\widehat{G}^{(h)}_{u} = -\frac{1}{h}AB^-\widehat{W}_{\gamma^-_+}. \tag{5.10}$$

In this treatment we perform the measurements at the layer γ^-_+ and set the AS sources at the layer γ^-. It is easy now to obtain the appropriate relation for the original time–dependent variables:

$$G^{(h)}_{u} = -\frac{1}{h}AB^-W_{\gamma^-_+}. \tag{5.11}$$

From the general AS solution (5.7) it follows that in the general case under the assumption of $kh << 1$ the AS solution is given by

$$G^{(h)}_{b} = \frac{1}{h}AW_{\gamma^-_+}. \tag{5.12}$$

Let us consider a quite general case of a self–conjugated matrix A. In particular, the acoustics equations can be represented via such a matrix. Then,

$$\left\|G^{(h)}_{b}\right\|^2_2 = \frac{1}{h^4}(AW, AW) \geqslant \frac{1}{h^4}(AB^-W, AB^-W) = \left\|G^{(h)}_{u}\right\|^2_2 \tag{5.13}$$

Along with the screening of the internal volume from the external field, we are able to consider screening the external field from the internal field. The consideration, similar to the analysis given above, leads us to the following results. Having neglected by the reflection from the boundary of the internal volume, the AS source term is given by

$$\widehat{G}^{(h)}_{u|\gamma^+} = -L_h V^{(h)}_{|\gamma^+} = (ik + A\frac{\nabla}{h})V^{(h)}_{\gamma^+} = (ik + \frac{1}{h}A)B^+\widehat{W}_{\gamma^+} \tag{5.14}$$

Here, we consider a layer γ^+_- that is the nearest layer to γ^+ in D ($m = -2$, in 1D case). Since, in turn, we take into account only the outcoming invariants, the vectors at γ^+ and γ^+_- are related via the ∇–approximation:

$$(ikh + \Lambda\nabla)\widehat{Y}_\gamma = 0$$

or

$$\widehat{Y}_{\gamma^+} = (\Lambda + ikhI)^{-1}\Lambda\widehat{Y}_{\gamma^+_-}. \tag{5.15}$$

Then, the source term is formulated via the vector W only at the layer γ^+_-:

$$\widehat{G}^{(h)}_{u|\gamma^+} = -\frac{1}{h}B^+A\widehat{W}_{\gamma^+_-} = -\frac{1}{h}AB^+\widehat{W}_{\gamma^+_-}. \tag{5.16}$$

As an example, let us apply the developed general results to the acoustics system of equations (4.1). In this case, the matrixes are as follows:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ S = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ B^- = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

Having set $\widehat{R}^-_{M^-\setminus\gamma} = 0$, we obtain:

$$g_{\gamma^-} = -1/2(ik + \frac{1}{h})\widehat{R}^-_0(1,-1)^T \tag{5.17}$$

This term coincides with the source term (4.13), (4.19) derived via the immediate Riemann–invariant analysis.

Using (5.10), the AS source term can be written via the value $\widehat{W}_1$:

$$g_{\gamma^-} = g_0 = -\frac{1}{h}B^-A\widehat{W}_1 = -\frac{1}{2h}\widehat{R}^-_1(1,-1)^T. \tag{5.18}$$

In the case of the external screening:

$$g_{\gamma^+_-} = g_{-1} = -\frac{1}{h}B^+A\widehat{W}_{-2} = -\frac{1}{2h}\widehat{R}^+_{-2}(1,\ 1)^T. \tag{5.19}$$

In the physical variables, the AS source terms can be represented in the following general form (see Nelson & Elliott (1992)):

$$\begin{aligned} p_t/c^2 + \rho u_x &= \rho q_{vol}, \\ \rho u_t + p_x &= f_{vol}, \end{aligned} \tag{5.20}$$

where q_{vol} is the volume velocity per a unit volume and f_{vol} is the force per a unit volume (see Nelson & Elliott (1992)).

From (4.19) and (5.18), one obtain that in the case of screening the domain D

$$q_{vol} = -\frac{1}{2h} R_1^-, \tag{5.21}$$

$$f_{vol} = \frac{1}{2h} R_1^-,$$

where $R^- = \frac{p}{\rho c} - u$. If the internal sources are absent and there is no reflection from the boundaries, then $R^- = 2\frac{p}{\rho c}$

It is easy to exclude one of the two variables in (5.20) and derive the equation for, e.g., the pressure:

$$-\frac{1}{c^2} p_{tt} + p_{xx} = g_p, \tag{5.22}$$

where

$$g_p = \frac{\partial}{\partial x} f_{vol} - \rho \frac{\partial}{\partial t} q_{vol}.$$

From (5.21), it follows that

$$g_p = \frac{\rho}{2h}(R_t^- + cR_x^-). \tag{5.23}$$

If the internal sources are absent and there is no reflection, then

$$g_p = \frac{2p_t}{hc}. \tag{5.24}$$

For the Fourier images of the pressure we have:

$$k^2 \widehat{p} + \widehat{p}_{xx} = \widehat{g}_p, \tag{5.25}$$

where

$$\widehat{g}_p = \frac{ik}{h} \widehat{R}_1^- = i\frac{2k}{h} \widehat{p}_1. \tag{5.26}$$

On the other hand, it is possible to consider the Helmholtz equation for the pressure immediately:

$$k^2 \widehat{p} + \widehat{p}_{xx} = 0. \tag{5.27}$$

Having approximated it, we obtain the following finite-difference equation:

$$k^2 \widehat{p}_m + \frac{1}{h^2}(\widehat{p}_{m+1} - 2\widehat{p}_m + \widehat{p}_{m-1}) = 0. \tag{5.28}$$

If the area $m < 0$ is shielded, the main theorem yields the following possible double–layer solution:

$$\widehat{g}_0^{(h)} = -(\frac{\widehat{p}_{-1} - 2\widehat{p}_0}{h^2} + k^2 \widehat{p}_0). \tag{5.29}$$

This expression includes the subtraction of two measured values $\widehat{p}_{-1}$ and $\widehat{p}_0$ and their division by h^2.

In Loncaric *et al.* (2003), it is shown that without the internal sources the local AS source term at the point $m = 0$ is given by

$$\widehat{q}_0^{(h)} = -\widehat{p}_0 (2\frac{q-1}{h^2} + k^2), \tag{5.30}$$

where $q = 1 - \frac{1}{2}k^2h^2 - ikh\sqrt{1-\frac{1}{4}k^2h^2}$. Thus,

$$\widehat{q}_0^{(h)} = i\frac{2k}{h}\widehat{p}_0\sqrt{1-\frac{1}{4}k^2h^2} \approx i\frac{2k}{h}\widehat{p}_0. \tag{5.31}$$

In the latest equality we assume that $kh << 1$ that means the mesh is fine enough to resolve the wave length.

The source terms $\widehat{g}_p$, given by (4.19), and $\widehat{q}_0^{(h)}$ approximately equal each other. In Loncaric *et al.* (2003) it is proved that the local source term (5.30) corresponds to the minimal source term in L_1 among the all possible AS solutions provided by the main theorem.

6. 3D statement

Let us now consider the fully 3D statement of the problem. The system of equations for the Fourier components can be written in the following form:

$$ik\widehat{W} + \sum_{j=1}^{3} A_j \widehat{W}_{x_j} = 0. \tag{6.1}$$

Suppose that system (6.1) is hyperbolic. Then,

$$A_j = S_j \Lambda_j S_j^{-1},$$

where the matrixes of eigenvalues $\Lambda_j = diag\{\lambda_k\}_j$ have only real diagonal elements.

The generalization of the scheme (5.5) is the following:

$$ik\widehat{W}_m + \sum_{j=1}^{3} \frac{1}{2h_j} A_j \dot{\Delta}\widehat{W}_m = \sum_{j=1}^{3} \frac{1}{2h_j} S_j |\Lambda_j| S_j^{-1} \Delta_2 \widehat{W}_m. \tag{6.2}$$

Unfortunately, in the general case system (6.1) cannot be reduced to a diagonal form. Therefore, the immediate generalization of the one–layer AS source terms (5.7), (5.8) is not possible. The general source term (4.9) in the general form is based on the double–layer measurements at γ. Nevertheless, one can derive an approximate local single–layer AS source term.

Let us rewrite system (6.2) in the orthogonal coordinate system $\{\xi^i\}$ related to the boundary of the domain shielded:

$$ik\widehat{W} + \sum_{k=1}^{3} \overline{A}_k \widehat{W}_{\xi_k} = 0, \tag{6.3}$$

where

$$\overline{A}_k = \sum_{j=1}^{3} \frac{\partial \xi^k}{\partial x^j} A_j.$$

Assume that the surface of the boundary corresponds to $\xi^3 = const$. System (6.3) can be approximated in the same way as the original system (6.1):

$$ik\widehat{W}_m + \sum_{j=1}^{3} \frac{1}{2h_j}\overline{A}_j \dot{\Delta}\widehat{W}_m = \sum_{j=1}^{3} \frac{1}{2h_j} S_j |\overline{\Lambda}_j| S_j^{-1} \Delta_2 \widehat{W}_m, \tag{6.4}$$

where

$$\overline{A}_j = \overline{S}_j \overline{\Lambda}_j \overline{S}_j^{-1}.$$

Having set $v^{(h)}_{|M^-\setminus\gamma} = 0$ in general solution (3.6), we obtain the double–layer distribution of the AS source terms on the layers γ^- and γ^-_+. It is important to note that the source terms on the layer γ^-_+ do not include the approximations of any derivatives but in the normal direction. On the other layer γ^-, the contribution of the approximations along ξ^1 and ξ^2 are $O(1)$ while the contribution of the normal derivative is $O(h_3^{-1})$. Thus, under the assumption of $h_3 \to 0$ only the normal derivative is remained. It leads us to a quasi 1D case.

Similarly to solution (5.12), under the assumption of $h_3 \to 0$ we have the following AS source term:

$$G^{(h)}_{b|\gamma^-} = \frac{1}{h_3}\overline{A_3} W_{\gamma^-}. \tag{6.5}$$

The optimum free–space solution is given by

$$G^{(h)-}_{u|\gamma^-} = -\frac{1}{h_3}\overline{A}_3 \overline{B}_3^{\,-} W_{\gamma^-_+}, \tag{6.6}$$

where

$$\overline{B}_3^{\,-} = \overline{S}_3 I^-_{\lambda,3} \overline{S}_3^{-1}, \tag{6.7}$$

$I^-_{\lambda,3}$ is a diagonal matrix having nonzero elements equaled to –1 if the appropriate eigenvalue of the matrix $\overline{B}_3^{\,-}$ is negative.

In the case of the acoustics equations

$$\begin{aligned} p_t + \rho c^2 \nabla \mathbf{u} &= \rho c^2 q_{vol}, \\ \mathbf{u}_t + \nabla p/\rho &= \mathbf{f}_{vol}, \end{aligned} \tag{6.8}$$

the matrix $\overline{A}_3$ is as follows:

$$\overline{A}_3 = \begin{pmatrix} 0 & n_1 & n_2 & n_3 \\ n_1 & 0 & 0 & 0 \\ n_2 & 0 & 0 & 0 \\ n_3 & 0 & 0 & 0 \end{pmatrix}, \tag{6.9}$$

where $n_j = \frac{\partial \xi^3}{\partial x^j}\,(j = 1,2,3)$ are the components of the normal to the boundary.

Then, the general AS source term under the assumption of $kh_3 << 1$ is given by

$$G^{(h)}_{b|\gamma^-} = h_3^{-1}\left(\rho c^2 v_n, \frac{p}{\rho} n_1, \frac{p}{\rho} n_2, \frac{p}{\rho} n_3\right)^T_{\gamma^-}, \tag{6.10}$$

where v_n is the normal component of the velocity. This means

$$q_{vol} = h_3^{-1} v_{n|\gamma^-}, \qquad \mathbf{b}_{vol} = h_3^{-1} p_{\gamma^-} \mathbf{n}. \tag{6.11}$$

The projection matrix $\overline{B}_3^-$ is represented as follows:

$$\overline{B}_3^- = \frac{1}{2} \begin{pmatrix} -1 & n_1 & n_2 & n_3 \\ n_1 & -n_{11} & -n_{12} & -n_{13} \\ n_2 & -n_{21} & -n_{22} & -n_{23} \\ n_3 & -n_{31} & -n_{32} & -n_{33} \end{pmatrix}, \tag{6.12}$$

where $n_{ij} = n_i n_j$.

Then, the optimal free-space AS source vector is given by:

$$G^{(h)}_{u|\gamma^-} = -\frac{1}{2} h_3^{-1} R^-_{\gamma_+^-} c(\rho c, -n_1, -n_2, -n_3)^T, \tag{6.13}$$

where $R^- = \frac{p}{\rho c} - v_n$. In this case

$$q_{vol} = -\frac{1}{2} h_3^{-1} R^-_{\gamma_+^-}, \qquad \mathbf{b}_{vol} = \frac{1}{2} h_3^{-1} R^-_{\gamma_+^-} \rho c \mathbf{n}. \tag{6.14}$$

If both the "friendly" sound and reflection from the shielded area are absent, then we can set $R^- = \frac{2p}{\rho c}$. In this case, the measurement of only one value (either the pressure or normal velocity) is required.

The obtained source term can be approximately used in a multidimensional case as a local 1D AS solution. In this case, the solution is considered in the direction normal to the boundary of the shielded domain and includes the normal component of the velocity. The more direction of the external wave front is close to the normal one the more solution is accurate. A similar case appears in the problem of non–reflecting boundary conditions when the local characteristic–type boundary conditions are used (see Tsynkov (1998)). In contrast to the non–reflecting boundary conditions, the local data obtained from the measurements allows us to reach a local solution either of single–layer or double–layer.

The same kind of AS solution is remained in the case of a broadband spectrum if the problem is linear. Indeed, the AS solution for Fourier components does not explicitly depend on both the frequency and phase. Since each frequency additively contributes to the measurement data, from the principle of superposition it follows that the source based on these data provides the AS.

7. Concluding remarks

The general single–layer solution of AS has been obtained for an arbitrary system of hyperbolic equations in a finite–difference form. The solution does not require either the knowledge of the Green's function or characteristics of the sources and medium. The AS can be realized via a single–layer source term requiring the measurements only at one layer nearby the domain screened. In the case of the domain shielded in a free space, the optimal single–layer solution has been obtained. The mutual location and influence of the AS source and measurement point has been considered. Single–layer and double–layer AS solutions have been obtained for a general 3D case.

8. Acknowledgment

This research was supported by the Engineering and Physics Sciences Research Council (EPSRC) under grant GR/26832/01 and the first author was also supported by the Russian Foundation for Basic Research (RFBR) under grant 05-01-00426A.

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