Notes on Proposition 1, Contagion and Volatility with Imperfect Credit Markets, IMF Staff Papers, June 1998.

The enclosed notes review the derivation of $\frac{\partial r_L}{\partial \delta_m}$; $\frac{\partial n}{\partial \delta_m}$, ($n$ is the employment of the representative producer, n_h in the IMF Staff Papers). The other results mentioned in Proposition 1 are obtained in a similar manner.

Assumptions:

- we operated in the range of partial default, as is depicted in Figure 4. This implies that $\delta_m \geq \delta *$. The equations in the paper fit this case.

- the country is at the threshold of full integration with the capital market, hence the probability of default is positive but small. It is easy to verify that the probability of default [area D in Figure 4] is $\frac{(\delta * + \delta_m)^2}{8\varepsilon_m \delta_m}$. Hence, we assume that $\delta * + \delta_m$ is relatively small.

To simplify the notation, we denote the first order condition determining employment [equation (13) in the IMF staff papers] by $H = 0$. The second order condition of optimization implies that $H_n' < 0$. Note also that $H_{r_L}' < 0$ (it is easy to verify that

$$H_{r_L}' = -C(1-\beta)\frac{\omega n^{-\beta}}{\kappa}[\frac{(\delta * + \delta_m) + \frac{\omega n}{\kappa n^{\beta}}(1 + r_L)}{4\varepsilon_m \delta_m}] < 0).$$

Figure A1 plots the equilibrium, where NN corresponds to H = 0 (see the end of this handout). Applying the above, NN is downwards sloping.

Bank's 'brake even' condition, [equation (8) in the IMF staff Papers] is denoted by $G = 0$, $G = \omega n(r_L - \tilde{r}_b^f) - \Gamma$, where

$$\text{(A1)} \quad \Gamma = \frac{\frac{\kappa n^{\beta}}{6}(\delta * + \delta_m)^3 + \frac{C}{2}(\delta * + \delta_m)^2}{4\varepsilon_m \delta_m}.$$

Equations (8) and (A1) imply

(A2) $$G'_{r_L} = \omega n - \frac{\frac{\kappa n^{\beta}}{2}(\delta^* + \delta_m)^2 + C(\delta^* + \delta_m)}{4\varepsilon_m \delta_m}\frac{\omega}{\kappa}n^{1-\beta}, \qquad G''_{r_L, r_L} = 0.$$

$$G'_n = \omega(r_L - \tilde{r}_b^f) - \frac{\frac{\kappa \beta n^{\beta-1}}{6}(\delta^* + \delta_m)^3 + \left[\frac{\kappa n^{\beta}}{2}(\delta^* + \delta_m)^2 + C(\delta^* + \delta_m)\right]\frac{\omega}{\kappa}(1-\beta)n^{-\beta}(1+r_L)}{4\varepsilon_m \delta_m} =$$

(A3) $$\frac{\Gamma}{n} - \frac{\frac{\kappa \beta n^{\beta-1}}{6}(\delta^* + \delta_m)^3 + \left[\frac{\kappa n^{\beta}}{2}(\delta^* + \delta_m)^2 + C(\delta^* + \delta_m)\right]\frac{\omega}{\kappa}(1-\beta)n^{-\beta}(1+r_L)}{4\varepsilon_m \delta_m};$$

$$G''_{n,n} > 0$$

Curve BB in Figure A1 corresponds to the brake even condition, $G = 0$. In general, this curve may be backward bending, implying the possibility of multiple equilibria, as are depicted in Figure A1 and A2. In these circumstances, the equilibrium with the lower interest rate is the efficient, leading to higher expected producer's surplus. Assuming rational expectation and a competitive equilibrium, producers will borrow from banks that will offer point I.[1] For example, in Figure A1, we will focus only on point I, along the upward sloping portion of BB, where $G'_{r_L} > 0$ and $G'_n < 0$, and we will ignore II.[2]

From $G = 0$ and $H = 0$, we infer that around point I

(A4) $$\begin{bmatrix} G'_n & G'_{r_L} \\ H'_n & H'_{r_L} \end{bmatrix}\begin{pmatrix} dn \\ dr_L \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} d\delta_m$$

1 For small probability of default, one can use (A2) and (A3) to show that $G'_{r_L} > 0$ and $G'_n < 0$.

2 Figure A2 represents another configuration, where the relevant equilibrium is at the low interest rate, where the probability of default is zero. The discussion in this Appendix (as well in proposition I) deals the configuration depicted by point I in Figure A1.

where $$\mu_1 = \frac{\partial \Gamma}{\partial \delta_m} = -\frac{\Gamma}{\delta_m} + \frac{\frac{\kappa n^\beta}{2}(\delta^* + \delta_m)^2 + C(\delta^* + \delta_m)}{4\varepsilon_m \delta_m}$$

$$= \frac{\kappa n^\beta (\delta^* + \delta_m)^2 (2\delta_m - \delta^*) + 0.5C(\delta^* + \delta_m)(\delta_m - \delta^*)}{4\varepsilon_m (\delta_m)^2} > 0$$

(the last inequality follows from $\delta_m \geq \delta^*$) , and

$$\mu_2 = -C(1-\beta)\omega n^{-\beta} \frac{1+r_L}{\kappa} \frac{\delta^*}{4\varepsilon_m (\delta_m)^2}.$$

Hence, (A4) $$\frac{\partial r_L}{\partial \delta_m} = \frac{\mu_2 G_n' - \mu_1 H_n'}{D}, \quad \frac{\partial n}{\partial \delta_m} = \frac{\mu_1 H_{r_L}' - \mu_2 G_{r_L}'}{D}$$, where $D = G_n' H_{r_L}' - H_n' G_{r_L}' > 0$

Note that for a small probability of default, $\delta^* + \delta_m \approx 0$; $\mu_1 \approx 0$; $\mu_2 > 0$; $G_n' < 0$; $G_{r_L}' \cong \omega n$.[3]

Hence,

(A5) $\frac{\partial r_L}{\partial \delta_m} > 0$; $\frac{\partial n}{\partial \delta_m} < 0$.[4]

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3 Note that $\delta^* + \delta_m \approx 0$ implies that $\delta^* \approx -\delta_m < 0$, hence $\mu_2 > 0$.

4 The result $\frac{\partial r_L}{\partial \delta_m} > 0$ is obtained by focusing on the terms proportional to $\delta^* + \delta_m$, ignoring higher order terms proportional to $(\delta^* + \delta_m)^2$ and $(\delta^* + \delta_m)^3$.

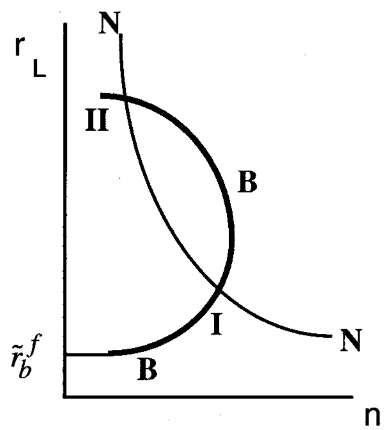

Figure A1

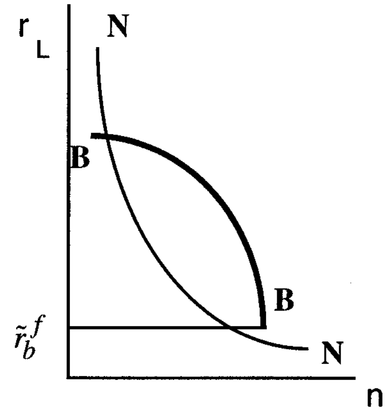

Figure A2