Simplicity of Nekrashevych algebras of contracting self-similar groups

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The free monoid X^*

X: set (alphabet)
X^*: the set of finite sequences (words) of elements of X
ε : the word of length 0

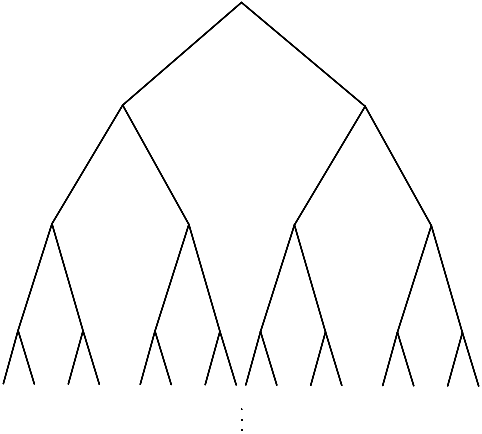
X^* is a monoid with respect to juxtaposition:

$$w \cdot v = wv$$

X^ω: the set of infinite sequences (words) of elements of X

Regular rooted trees

X: finite set with $|X| \geq 2$, T_X: $|X|$-regular rooted tree



vertices $\longleftrightarrow X^*$

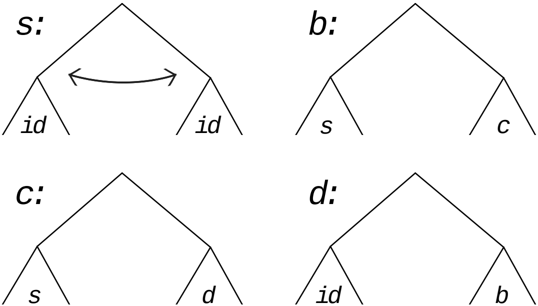
$\mathrm{Aut}(T_X)$:
automorphisms of T_X

$\mathrm{Aut}(T_X) \hookrightarrow S_{X^*}$
Let $g \in \mathrm{Aut}(T_X)$.

- $|g(w)| = |w|$ for all $w \in X^*$
- $g|_w \in \mathrm{Aut}(T_X)$: the restriction of g to the subtree at w
- $g(wu) = g(w)g|_w(u)$

The Grigorchuk group

The Grigorchuk group is generated by the following elements:



Contracting groups

A self-similar group $G \leq \mathrm{Aut}(T_X)$ is called **contracting** if there exists a finite set $N \subseteq G$ such that

for all $g \in G$ there exists k with $g|_w \in N$ whenever $|w| \geq k$.

The minimal such N is called the **nucleus**.

Contracting self-similar groups include a lot of the famous classes (such as the Grigorchuk group and the Gupta-Sidki p-groups).

If G is contracting, and generated by a finite set closed under sections, then there is an algorithm that computes N.

The Nekrashevych algebra

G: self-similar group, X: set with $|X| > 2$.

The Nekrashevych algebra $N_K(G, X)$ is the K-algebra generated by the set G and $\{x, x^* : x \in X\}$, subject to the relations

- $g \cdot h = gh$,
- $gx = g(x)g|_x$,
- $x^*g = g|_{g^{-1}(x)}(g^{-1}(x))^*$,
- $x^*y = \delta_{x,y}$,
- $\sum_{x \in X} xx^* = 1$.

Nekrashevych first defined a C^*-algebra defined by these relations, and later studied this discrete counterpart.
If G is contracting, G can be replaced by N, and $N_K(G, X)$ is finitely presented.

Nekrashevych algebras and inverse semigroups

S: monoid with zero, generated by G and $\{x, x^* : x \in X^*\}$, subject to the relations

- $g \cdot h = gh$,
- $gx = g(x)g|_x$,
- $x^*g = g|_{g^{-1}(x)}(g^{-1}(x))^*$,
- $x^*y = \delta_{x,y}$.

S is called the inverse semigroup associated to G,

$$S = \{ugv^* : g \in G, u, v \in X^*\} \cup \{0\}.$$

$N_K(G, X)$ in the Steinberg algebra of the tight groupoid of S, or algebraically,

$$N_K(G, X) = K_0S/(1 - \sum_{x \in X} xx^*),$$

where $(1 - \sum_{x \in X} xx^*)$ is the tight ideal $T_K(S)$ of S.

Simplicity of Nekrashevych algebras

The question we were interested in is

when is $N_K(G, X)$ simple?

$N_{\mathbb{C}}(G, X)$ not simple $\Longrightarrow$ the C^*-algebra is not simple

- Brown, Clark, Farthing, Sims (2015): if the ample groupoid associated to G is Hausdorff, $N_K(G, X)$ is simple for any K
- Nekrashevych (2015): the Nekrashevych algebra of the Grigorchuk group is not simple over characteristic 2
- Clark, Exel, Pardo, Sims, Starling (2018): characterize simplicity in terms of the groupoid, and prove that the Nekrashevych algebra of the Grigorchuk group is simple over all other characteristics
- Nekrashevych (2019): the Nekrashevych algebra of the Grigorchuk-Erschler group is simple over no field

Main results

Theorem (Steinberg, Sz.)

Let G be a contracting group with nucleus N.

- *Either $N_K(G, S)$ is simple over no field or simple over all but finitely many positive characteristics.*
- *There is an algorithm which on input N outputs the set of characteristics over which $N_K(G, S)$ is non-simple.*

The result gives a hands-on description of simplicity for several well-known infinite families of contracting self-similar groups.

For any finite set of primes $\mathcal{P}$, we give a contracting self-similar group G such that $N_K(G, X)$ fails to be simple exactly over characteristics in $\mathcal{P}$.

A nice characterization

$N_K(G,X)$ can be represented on the vector space KX^ω:
For $x \in X$ and $w \in X^\omega$ we can define

$$x \cdot w = xw,$$

$$x^* \cdot yw = \begin{cases} w & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

and any $g \in G$ acts on infinite words by a natural extension of the action on X^*.

Theorem (Steinberg, Sz.)

$N_K(G,X)$ is simple if and only if its representation on KX^ω is faithful. Moreover, the image of the representation is the unique simple quotient of $N_K(G,X)$.

Simplicity of inverse semigroup algebras

Theorem (Steinberg, Sz.)

*If S is a congruence-free inverse semigroup, then there is a unique maximal ideal of $K_0 S$ containing $T_K(S)$, called the **singular ideal** $I_K(S)$.*

- $K_0 S / T_K(S)$ has a unique maximal ideal: $I_K(S)/T_K(S)$.
- $K_0 S / T_K(S)$ is simple $\iff I_K(S) = T_K(S)$

If G is self-similar over X, then the associated semigroup S is congruence-free. The singular ideal of $K_0 S$ is

$$I_K(S) = \{a \in K_0 S : \text{for all } u \in X^* \text{ there is } w \in X^* \text{ with } auw = 0\}.$$

So the unique maximal ideal of $N_K(G, X)$ is

$$I_K(S)/(1 - \sum_{x \in X} xx^*).$$

The singular ideal minus the tight ideal

Let G be a self-similar over X and S the inverse semigroup. We have established:

$$N_K(G,X) \text{ is simple} \iff I_K(S) \setminus (1 - \sum_{x \in X} xx^*) = \emptyset.$$

$I_K(S) = \{a \in K_0 S : \text{for all } u \in X^* \text{ there is } w \in X^* \text{ with } auw = 0\}.$

Step 1:
For any $a \in K_0 S$, we have $a \in (1 - \sum_{x \in X} xx^*)$ if there are finitely many words $w \in X^*$ with $aw \neq 0$.

Step 2:
If $I_K(S) \setminus (1 - \sum_{x \in X} xx^*) \neq \emptyset$, then $I_K(S) \setminus (1 - \sum_{x \in X} xx^*)$ intersects KG. If G is contracting with nucleus N, it intersects KN.

Understanding $aw = 0$

Let $a = \sum_{g \in N} a_g g \in KN$. Then

$$aw = \sum_{g \in N} a_g gw.$$

$\equiv_w$: equivalence on N defined by $g \equiv_w h$ iff $gw = hw$. Then

$$aw = 0 \iff \forall h \in N, \sum_{g \equiv_w h} a_g = 0$$

For any equivalence $\equiv$ on N, consider the following linear system in variables $x_g, g \in N$:

$$\sum_{g \equiv h} x_g = 0,\ \ h \in N.$$

We say $a \in KN$ satisfies $\equiv$ if $a_g = x_g$ is a solution over K.

Understanding $aw = 0$

N is finite $\Longrightarrow$ there are finitely many equivalences and corresponding linear systems.

The key: to understand

- which of these must be satisfied for a to be singular,
- and which of these must not be satisfied for $a \notin (1 - \sum_{x \in X} xx^*)$.

The simplicity graph

Recall: $gx = g(x)g|_x$ for $g \in G$, $x \in X$, furthermore, if $g \in N$, then $g|_x \in N$.

So $g \equiv_{xw} h \iff gxw = hxw \iff g(x)g|_x w = h(x)h|_x w \iff g(x) = h(x)$ and $g|_x w = h|_x w \iff g(x) = h(x)$ and $g|_x \equiv_w h|_x$.

X^* has a left action on equivalences on N given by

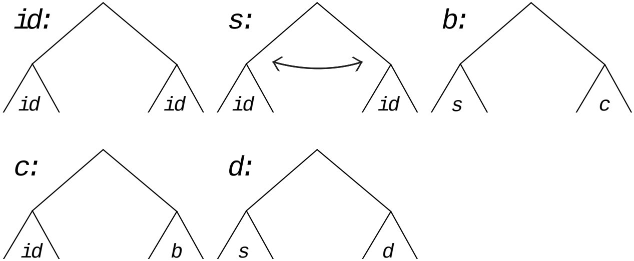
$$g\ x\cdot \equiv\ h \iff g(x) = h(x) \text{ and } g|_x \equiv h|_x.$$

$\{\equiv_w : w \in X^*\}$ is the orbit of the equality.

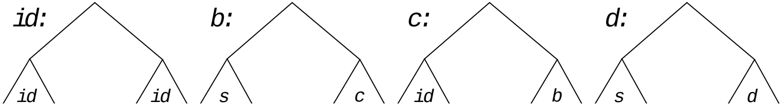
The simplicity graph: the Schreier graph of the action on $\{\equiv_w : w \in X^*\}$

An example: the Grigorchuk-Erschler group

The nucleus (also a generating set):



Note: $s(w) \neq g(w)$ if $s \neq g$, so s is its own $\equiv_w$-class for any w.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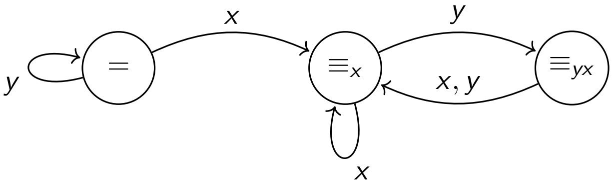
y
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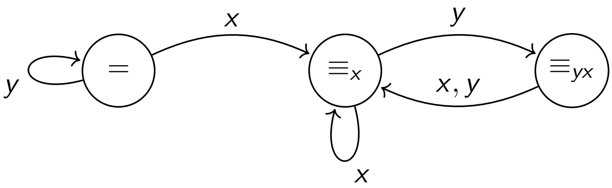
Reading the ideals from the simplicity graph

- $a \notin (1 - \sum_{x \in X} xx^*) \iff$ there is an equivalence $\equiv$ not satisfied by a, reachable from a cycle.
- $a \in I_K(S) \iff$ satisfies all the equations in the minimal strongly connected components.

The Grigorchuk-Erschler group



The minimal component:
$\equiv_x$: $\{s\}, \{b, d\}, \{id, c\}$
$\equiv_{yx}$: $\{s\}, \{c, d\}, \{id, b\}$

The equations:
$x_s = 0, x_d = -x_b, x_{\text{id}} = -x_c$
$x_s = 0, x_d = -x_c, x_{\text{id}} = -x_b$

$a = \text{id} - b - c + d$ is a solution over any field $\implies a \in I_K(S)$
a does not satisfy '$=$' which is reachable from a cycle, so
$a \notin (1 - \sum_{x \in X} xx^*)$
$\implies$ the Nekrashevych algebra is simple over no field.

Thanks for your attention!