

# Singular functions in groupoid algebras

Newcastle, Operator Algebra Seminar

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Nóra Szakács

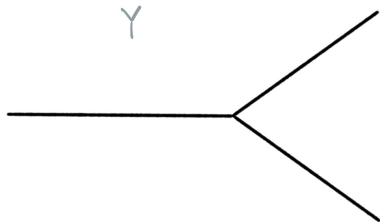
University of Manchester

March 3 2026

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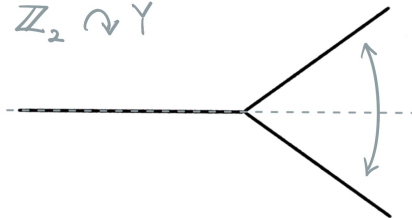


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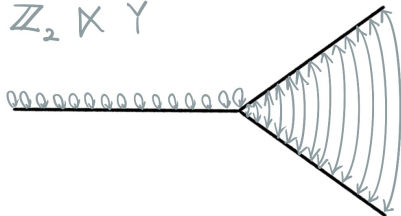
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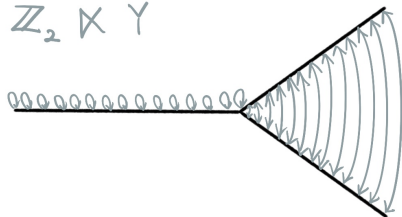
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**Morphisms:**

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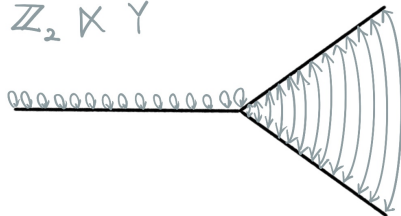
$$\{(g, y) : y \in Y, g \in G\}$$

$$s(g, y) = y, r(g, y) = g(y),$$

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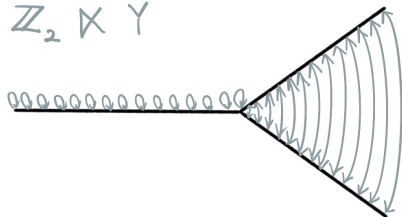
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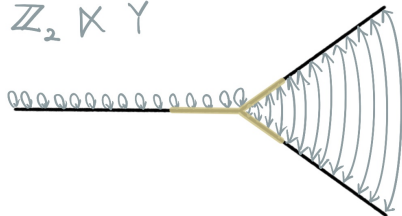
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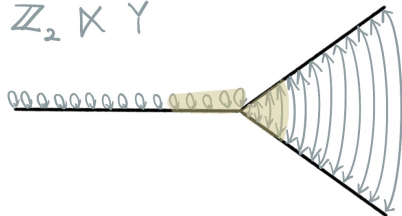
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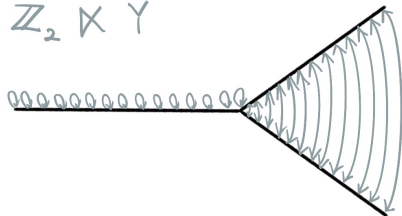
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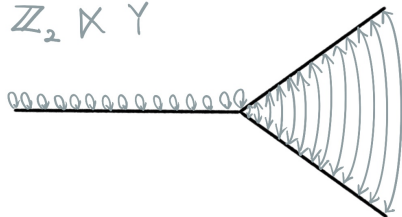
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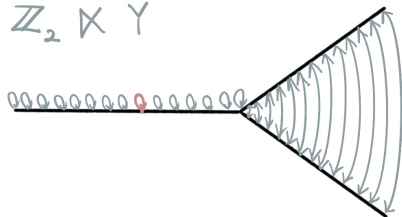
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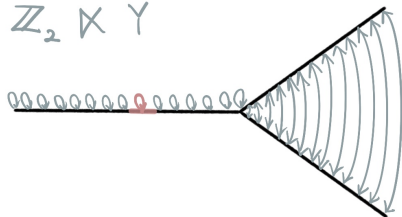
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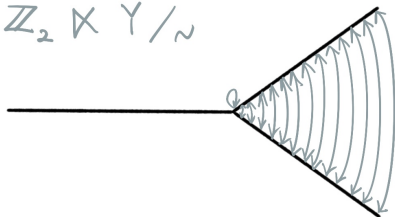
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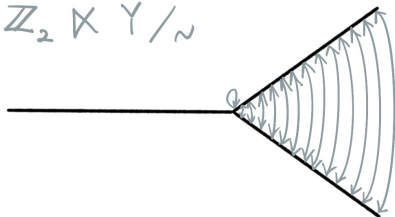
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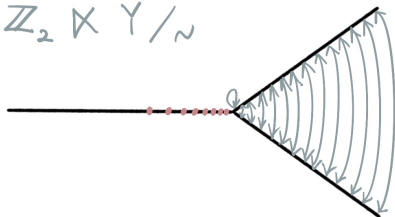
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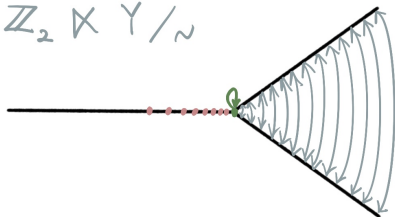
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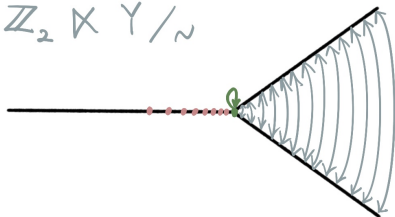
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It is typically *not* a Hausdorff space!

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# Étale groupoids

Let  $\mathcal{G}$  be a topological groupoid, i.e. a groupoid with a topology such that source and range maps, and the operations are continuous.

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$\mathcal{G}$  is **ample** if:

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$\mathcal{G}$  is **ample** if:

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Given an ample groupoid  $\mathcal{G}$ , its (complex) Steinberg algebra  $\mathbb{C}\mathcal{G}$  consist of

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**Example:** let  $f \in \mathbb{C}\mathcal{G}$  and  $U \subseteq \mathcal{G}^{(0)}$  compact open.

Then

$$f * \chi_U(\gamma) = \begin{cases} f(\gamma) & \text{if } s(\gamma) \in U, \\ 0 & \text{otherwise.} \end{cases}$$

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**Remark:**  $\mathbb{C}\mathcal{G}$  also recovers inverse semigroup algebras.

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Then  $\lambda(\chi_g)(\delta_h) = \sum_{k \in G} \chi_g(k)\delta_{kh} = \delta_{gh}$ , so  $\lambda$  is the usual representation of  $\mathbb{C}G$  on  $\mathcal{B}(\ell^2(G))$ , and  $C_{red}^*(\mathcal{G}) \cong C_{red}^*(G)$ .

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## **Theorem (Renault, '80)**

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From now on we identify  $\mathfrak{a}$  with  $j(\mathfrak{a})$  in notation.

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**Remark:** any groupoid of germs (as we defined it) is effective

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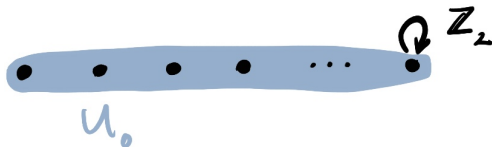
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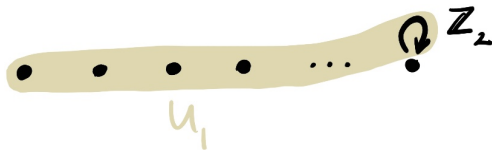
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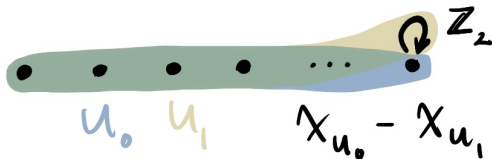
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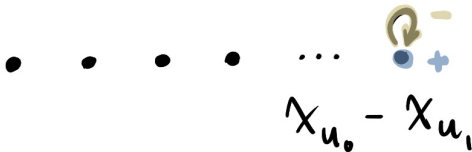
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It is natural to ask:

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# The strategy

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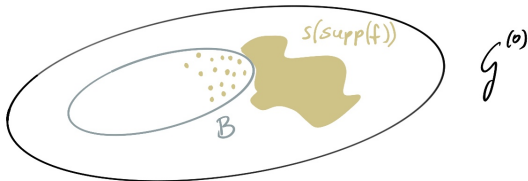
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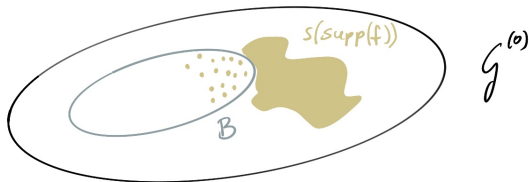
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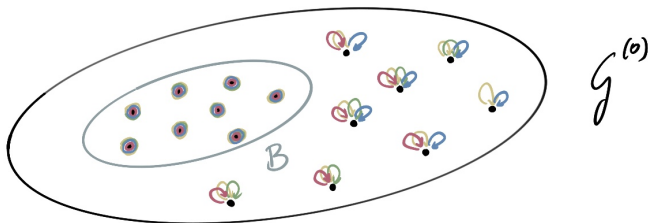
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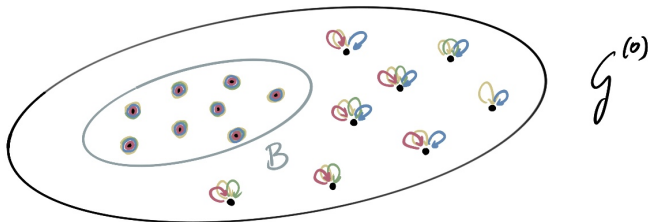


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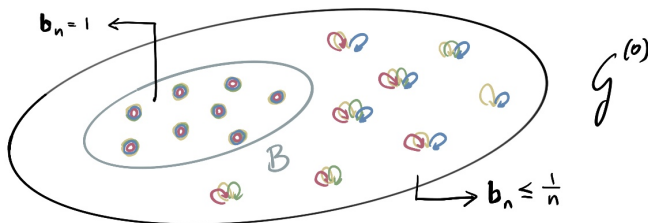


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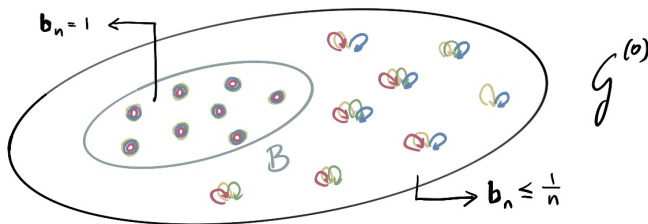


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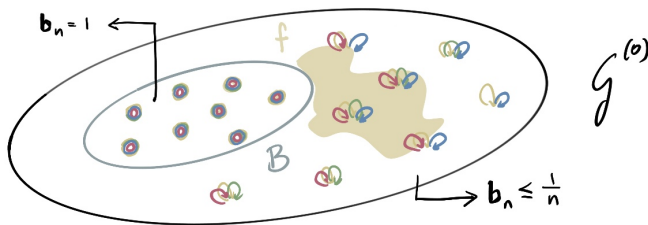


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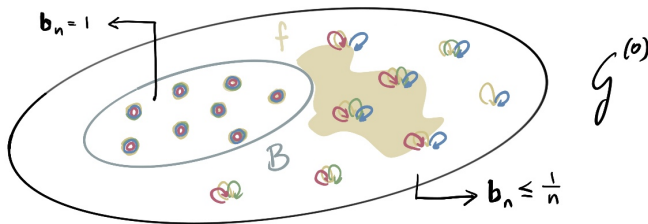
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What about convergence in norm?

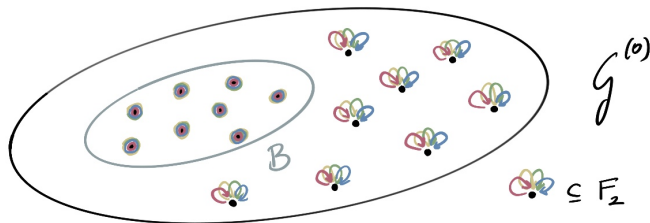
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**Need:**  $b_n = \frac{1}{n} \sum_{i=1}^n \chi_{B_i}$  converges in norm; i.e.  $b_n|_{\mathcal{G}_{B^c}} \xrightarrow{\|\cdot\|} 0$ .

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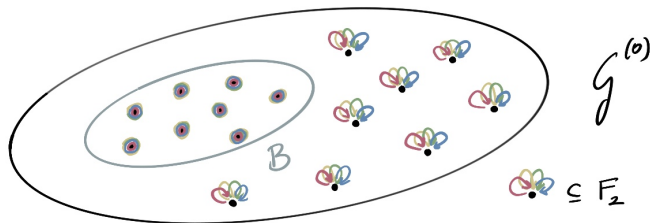
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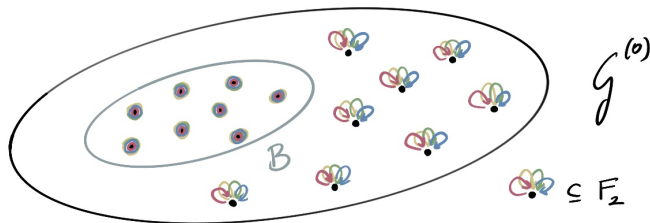
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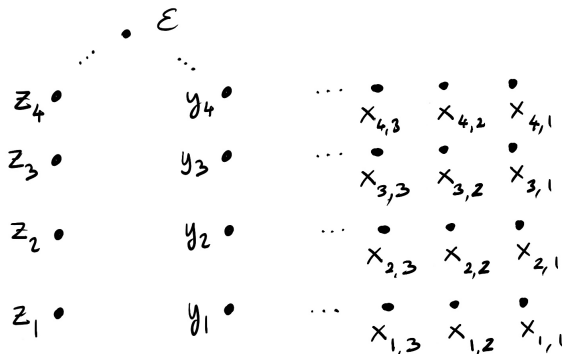
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**Note:** non-amenability is necessary here: in any amenable group,

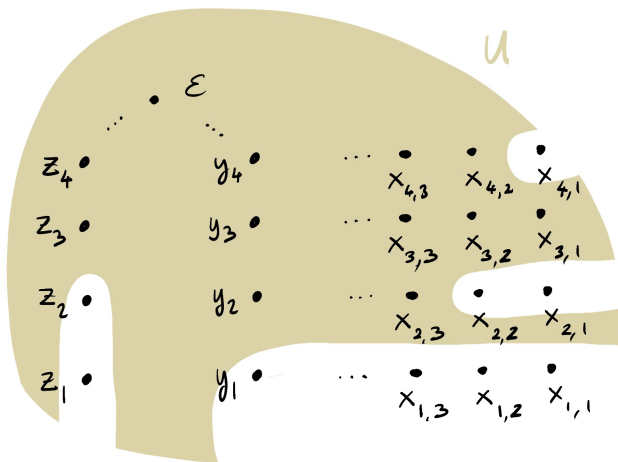
$$1 \leq \left\| \frac{1}{n} \sum_{i=1}^n g_i \right\|.$$

# The example: the definition of $\mathcal{G}$



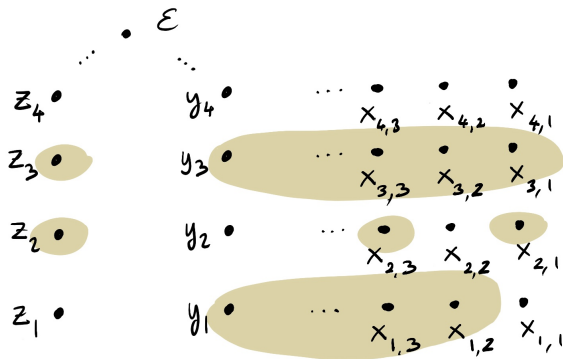
$\mathcal{G}^{(0)}$ : black dots above (where  $\dots$  denotes convergence)

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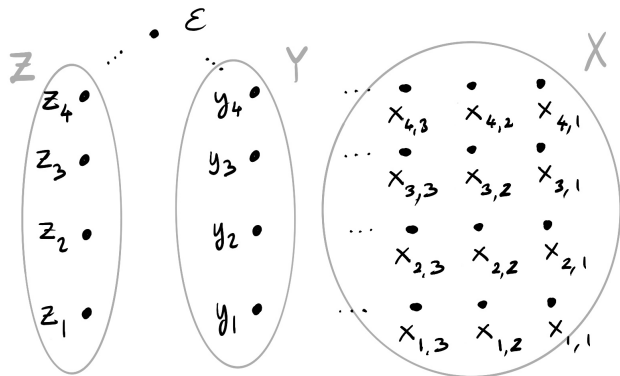
A 'typical' basic compact open neighborhood of  $\epsilon$

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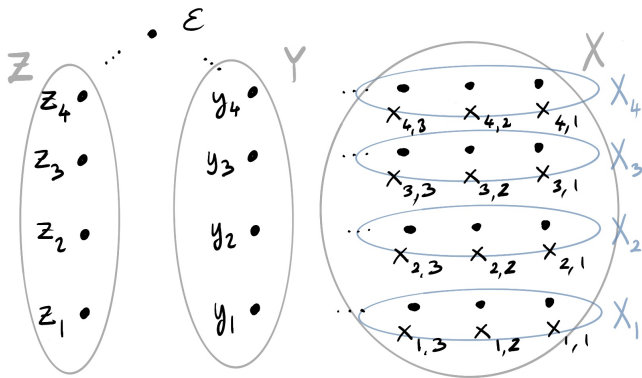
A 'typical' basic compact open set not containing  $\epsilon$

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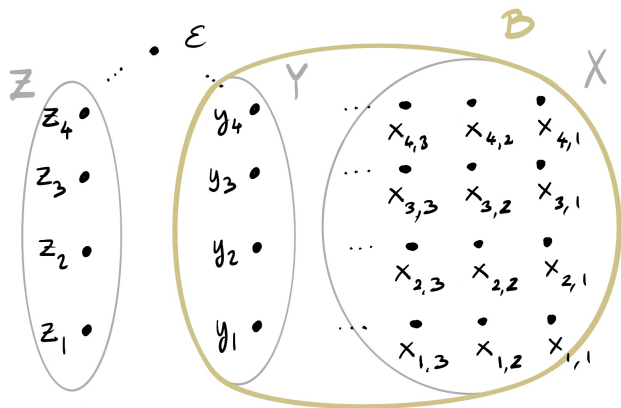
Notation:  $X, Y, Z$

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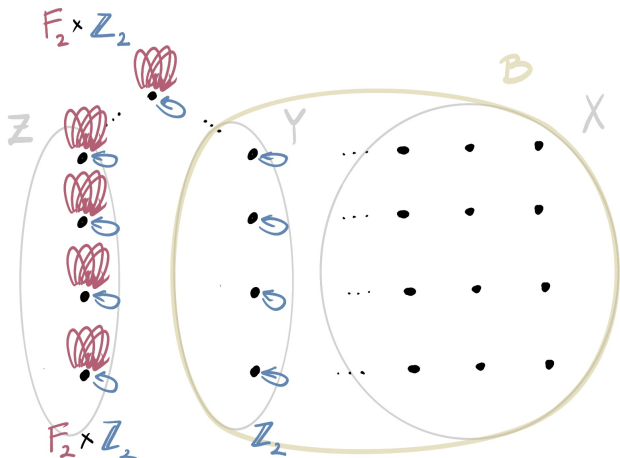
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**Notation:**  $X$ ,  $Y$ ,  $Z$  and  $B$ : the set we 'scatter' to

## The example: the definition of $\mathcal{G}$



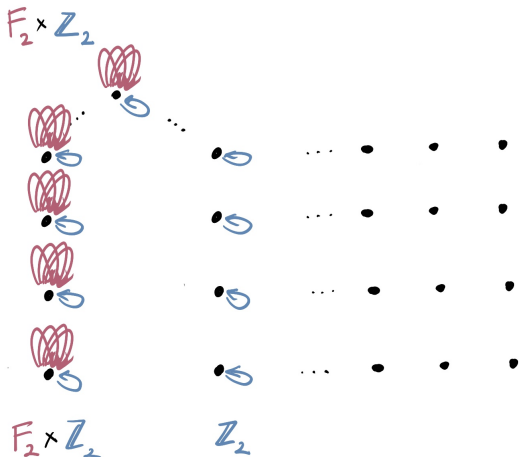
### The groupoid:

units in  $Z$  and  $\epsilon$  have the isotropy group  $F_2 \times \mathbb{Z}_2$

units in  $Y$  have the isotropy group  $\mathbb{Z}_2$

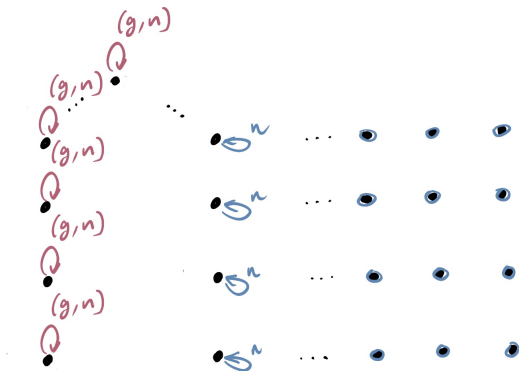
units in  $X$  have trivial isotropy group

# The example: the definition of $\mathcal{G}$



Basic compact open bisections:

## The example: the definition of $\mathcal{G}$

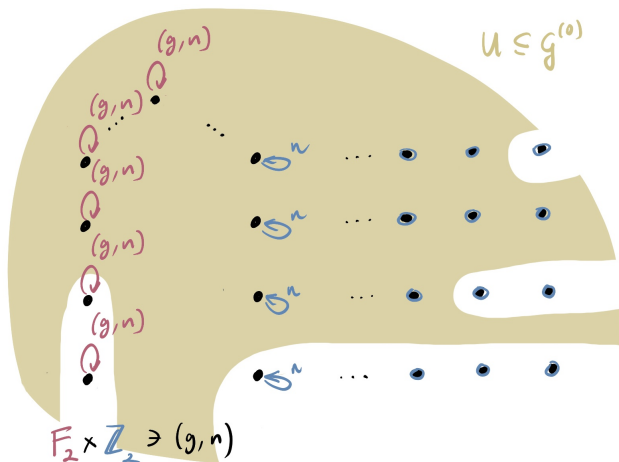


$$F_2 \times \mathbb{Z}_2 \ni (g, n)$$

**Basic compact open bisections:**

Every pair  $(g, n) \in F_2 \times \mathbb{Z}_2$  defines a compact open bisection

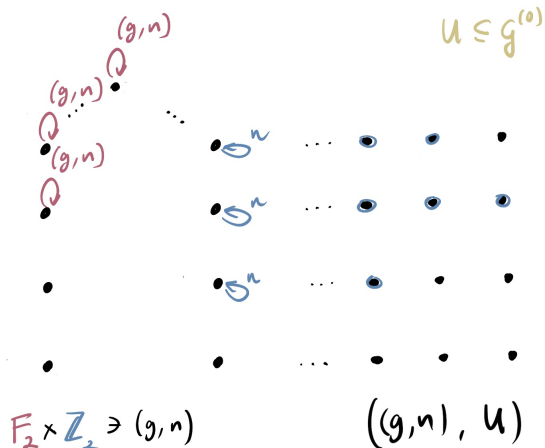
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Every pair  $(g, n) \in F_2 \times \mathbb{Z}_2$  defines a compact open bisection  
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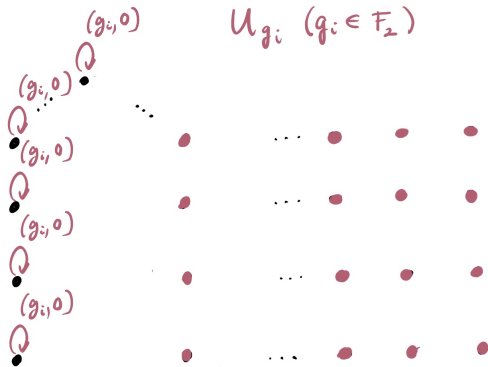
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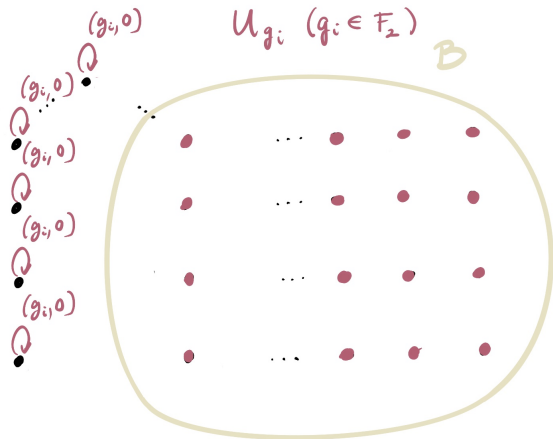
# The example: the singular element of $C_{red}^*(\mathcal{G})$



Let  $g_i \in F_2$  be such that  $\frac{1}{n} \sum_{i=1}^n g_i \rightarrow 0$  in  $C_{red}^*(F_2)$ .

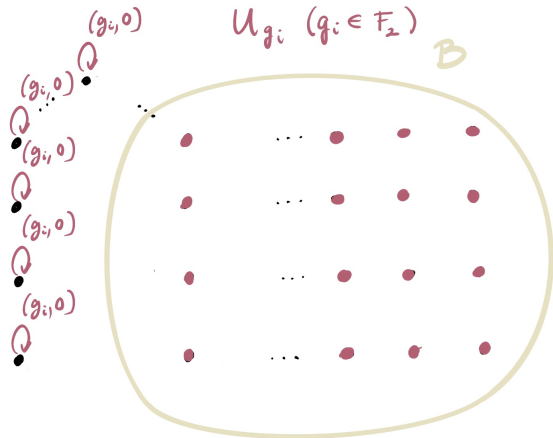
Consider the compact open bisection  $U_{g_i} = ((g_i, 0), \mathcal{G}^{(0)})$ .

# The example: the singular element of $C_{red}^*(\mathcal{G})$



$$\text{We 'scatter' to } B: \mathfrak{b}_n = \frac{1}{n} \sum_{i=1}^n \chi_{U_{g_i}} \xrightarrow{\|\cdot\|_\infty} \chi_B$$

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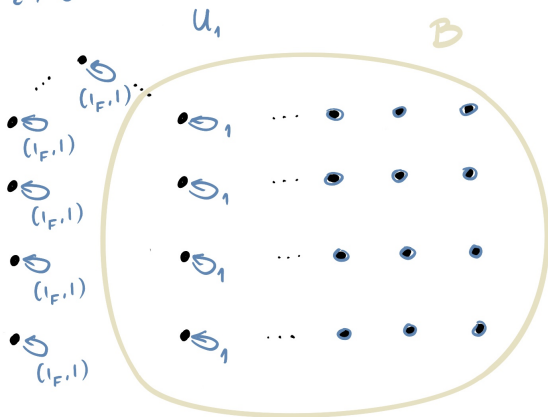


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The choice of  $g_i$ s ensures convergence in  $C^*$ -norm

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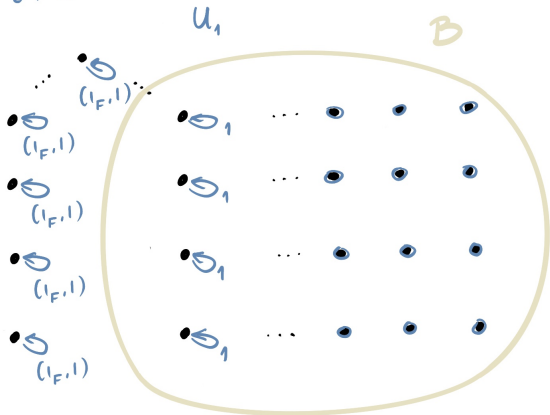
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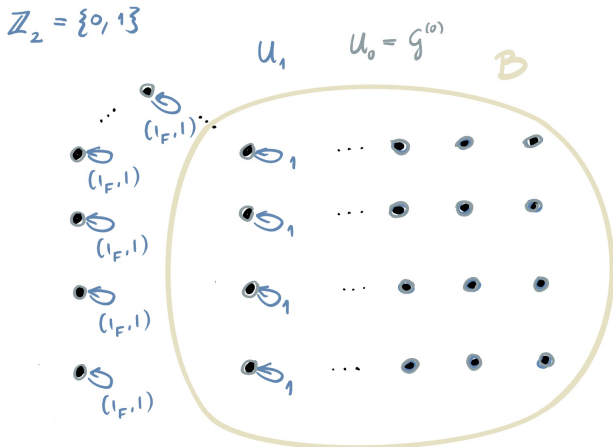


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Let  $U_1 = ((1_F, 1), \mathcal{G}^{(0)})$  compact open bisection

Let  $U_0 = \mathcal{G}^{(0)}$  compact open bisection

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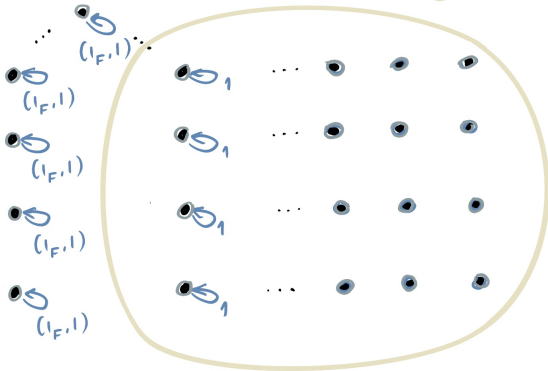
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$$\chi_{u_1} - \chi_{u_0}$$

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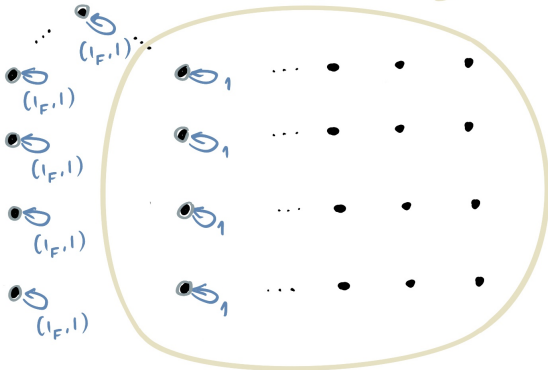
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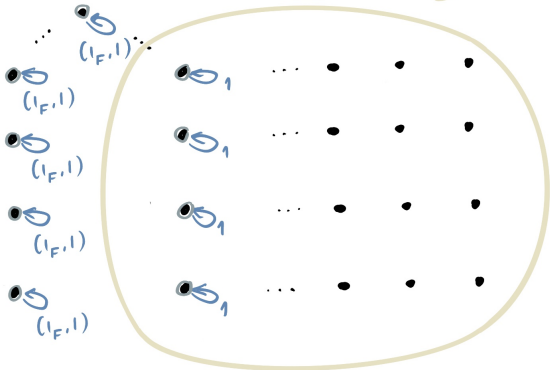
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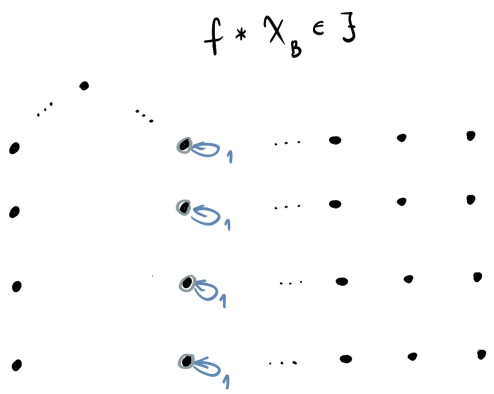


We create a  $B$ -singular function:

$$f = \chi_{U_1} - \chi_{U_0}$$

$\text{supp}(f)|_B$  contains no open sets  $\implies f * \chi_B \in J$

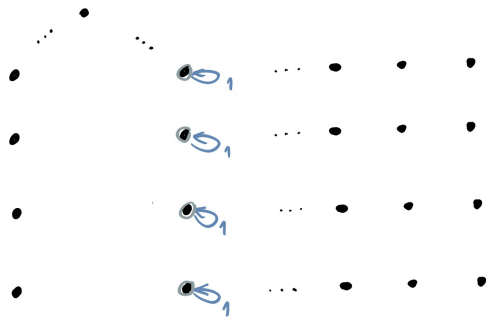
The example:  $f * \chi_B \notin \overline{J_{alg}}$



What remains is to show that  $f * \chi_B \notin \overline{J_{alg}}$ .

The example:  $f * \chi_B \notin \overline{J_{alg}}$

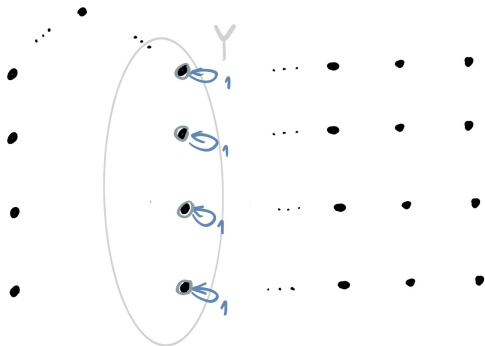
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We show something stronger:  $f * \chi_B \notin \overline{J_{alg}^\infty}$ .

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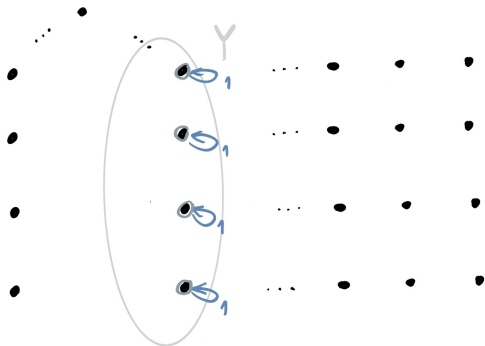
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Observe that if  $\|f - g\|_\infty < 1$  then  $Y \in \text{supp } g$ .

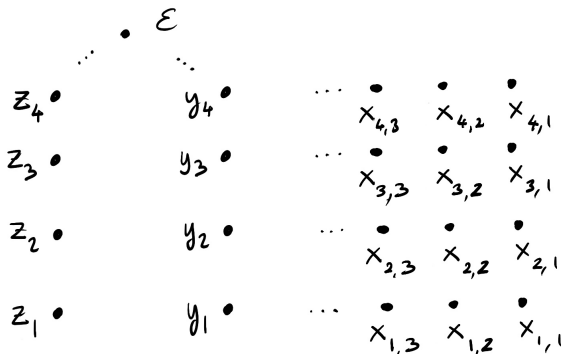
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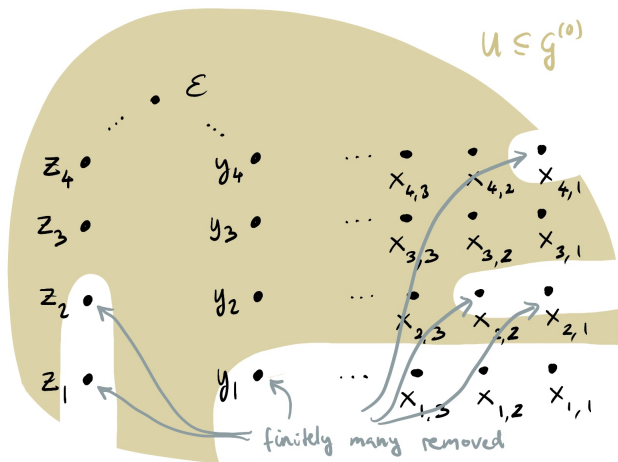
We show something stronger:  $f * \chi_B \notin \overline{J_{alg}}^\infty$ .  
 Observe that if  $\|f - g\|_\infty < 1$  then  $Y \in \text{supp } g$ .  
**Claim:** if  $Y \subseteq \text{supp } g$  for  $g \in \mathbb{C}\mathcal{G}$ , then  $g \notin J_{alg}$ .

The example:  $f * \chi_B \notin \overline{J_{alg}}$



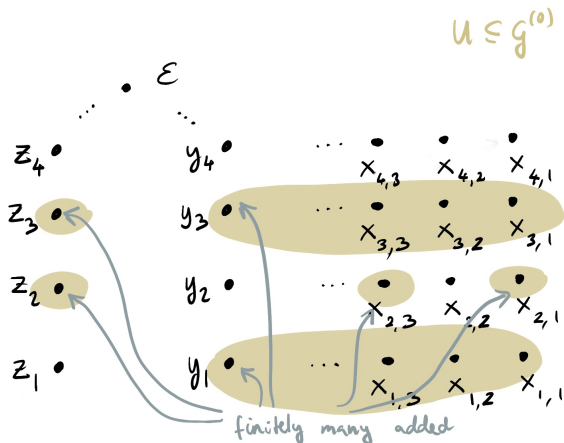
Let  $g = \sum_{V \in F} c_V \chi_V$  where  $F$  is a finite set of compact open bisections, and suppose  $Y \subseteq \text{supp } g$ .

The example:  $f * \chi_B \notin \overline{J_{alg}}$



Any compact open bisection  $V \in F$  is of the form  $((g, n), U)$  where  
 (1)  $U$  is either  $G^0$  with finitely many sets  $\{x_{i,j}\}, \{y_k\} \cup X_k, \{z_l\}$  'removed'

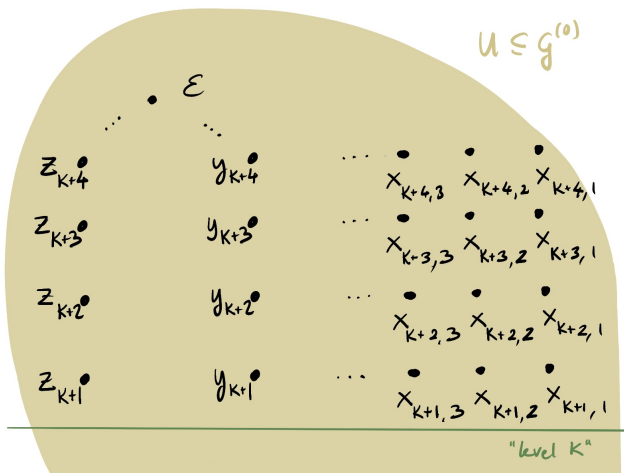
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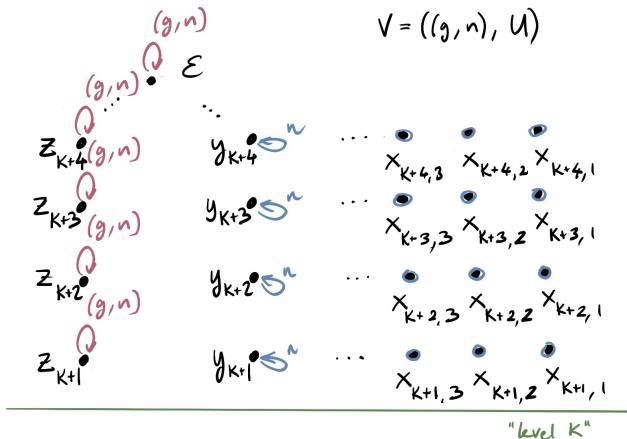
- (1)  $U$  is either  $G^0$  with finitely many sets  $\{x_{i,j}\}, \{y_k\} \cup X_k, \{z_l\}$  'removed'
- (2) or  $U$  consists of finitely many sets  $\{x_{i,j}\}, \{y_k\} \cup X_k, \{z_l\}$  'added'

The example:  $f * \chi_B \notin \overline{J_{alg}}$



Since  $F$  is finite, there exists some  $K \in \mathbb{N}$  larger than any 'added' or 'removed' indices

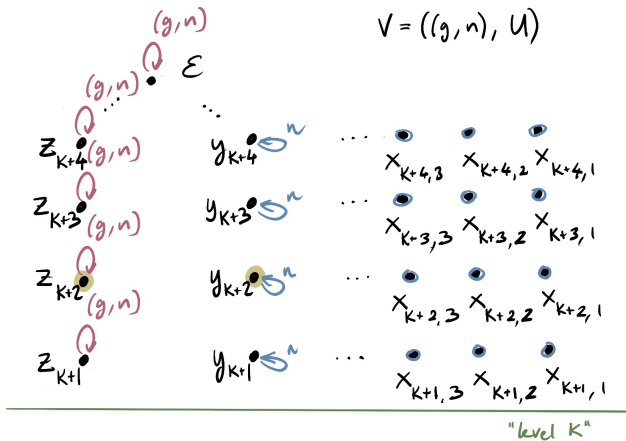
The example:  $f * \chi_B \notin \overline{J_{alg}}$



Since  $F$  is finite, there exists some  $K \in \mathbb{N}$  larger than any 'added' or 'removed' indices

Above the 'Kth level', any  $V \in F$  looks either empty (type (ii)) or a like a full bisection (type(i)).

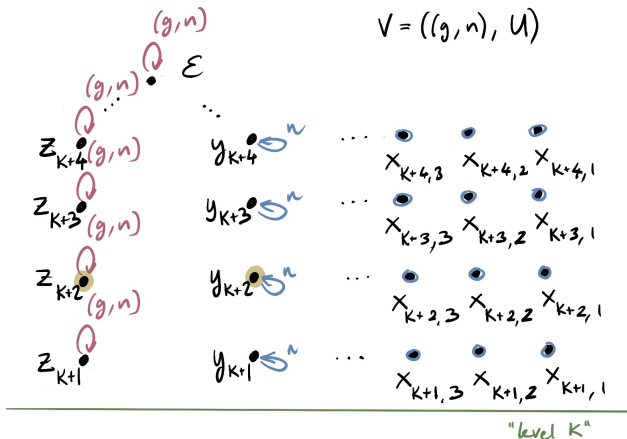
The example:  $f * \chi_B \notin \overline{J_{alg}}$



Recall:  $g = \sum_{V \in F} c_V \chi_V$ . Let  $i \geq K$ . Then

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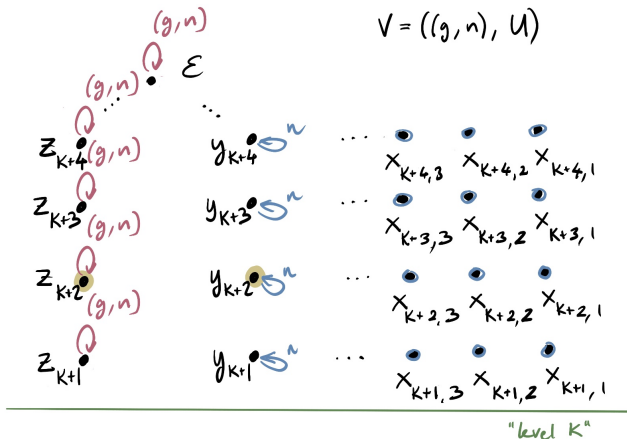
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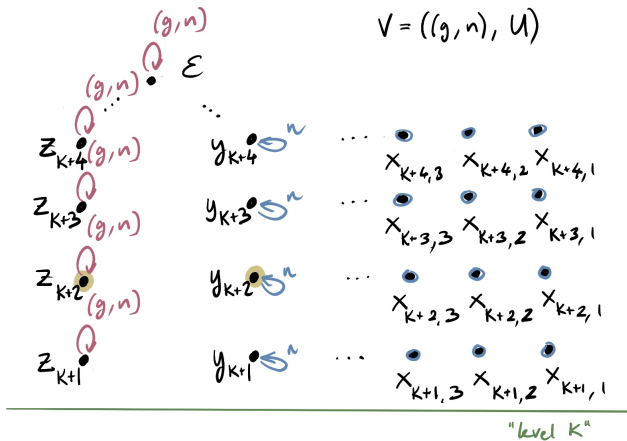
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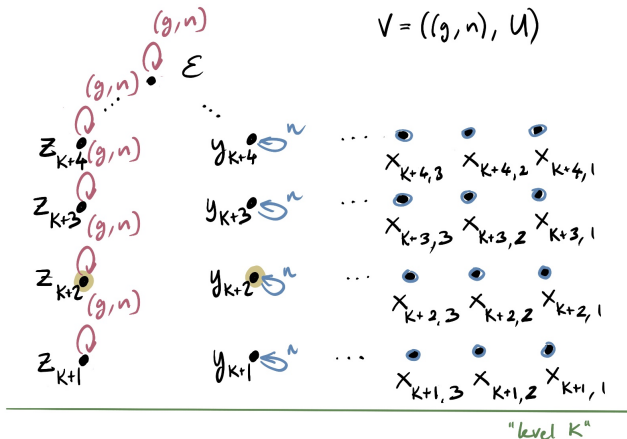
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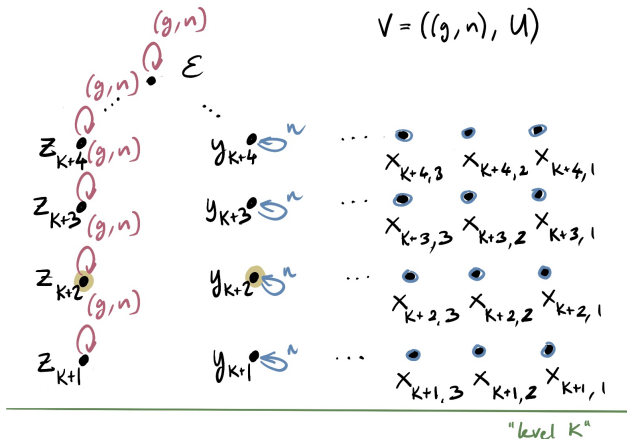
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This completes the proof that  $f * \chi_B \notin \overline{J_{alg}}$ .

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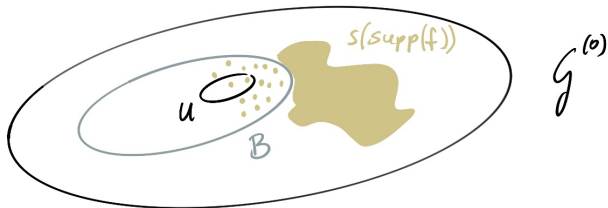
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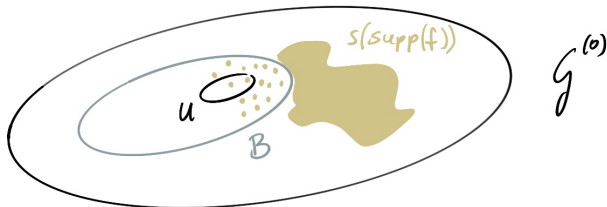
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In particular, this is still open:

**Question 1:** Does  $J_{alg} = 0$  imply  $J = 0$ ?