Simplicity of Nekrashevych algebras of contracting self-similar groups

Nóra Szakács
co-author: Benjamin Steinberg

University of Manchester

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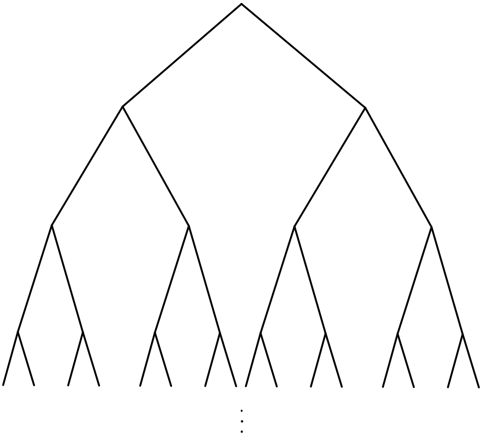
Letters and words

X: set (alphabet)
X^*: the set of finite sequences (words) of elements of X
ε : the word of length 0
X^ω: the set of infinite sequences (words) of elements of X

Rooted trees

X: finite set with $|X| \geq 2$, T_X: $|X|$-regular rooted tree



vertices $\longleftrightarrow X^*$

$\mathrm{Aut}(T_X)$:
automorphisms of T_X

$\mathrm{Aut}(T_X) \hookrightarrow S_{X^*}$ Let
$g \in \mathrm{Aut}(T_X)$.

- $|g(w)| = |w|$ for all $w \in X^*$
- $g|_w \in \mathrm{Aut}(T_X)$: the restriction of g to the subtree at w
- $g(wu) = g(w)g|_w(u)$

Self-similar groups

$G \leq \mathrm{Aut}(T_X)$ is **self-similar** if $g|_w \in G$ for all $g \in G, w \in X^*$.

Self-similar groups gained popularity in the 80's for yielding easy-to-define examples with exotic properties, such as:

- the first f.g. group of intermediate growth (Grigorchuk's group),
- easily understood infinite p-groups (Gupta-Sidki groups),

and were later linked to dynamical systems via iterated monodromy groups introduced by Nekrashevych.

The odometer

Let $X = \{0, 1\}$ and $g \in \mathrm{Aut}(T_X)$ the automorphism defined by

$$g(0) = 1, g|_0 = \mathrm{id}, g(1) = 0, g|_1 = g.$$

For example

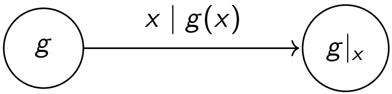
$g(0101) = g(0)g|_0(101) = 1101;$

$g(1101) = g(1)g|_1(101) = 0g(101) = 0g(1)g|_1(01) = 00g(01) = 00g(0)g|_0(1) = 0011.$

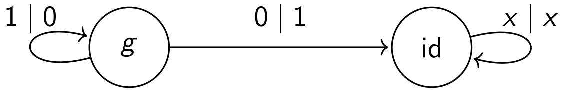
The **odometer** is the self-similar group $\langle g \rangle \leq \mathrm{Aut}(T_X)$, it encodes binary addition and is isomrorphic to $\mathbb{Z}$.

The state diagram

Given any set $S \subseteq \mathrm{Aut}(T_X)$ closed under taking sections, we can encode S by an edge-labeled digraph with vertex set S and edges

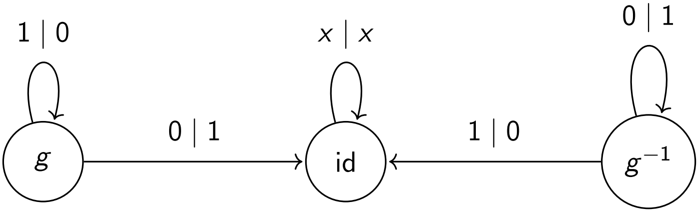


In the odometer, for $S = \{\mathrm{id}, g\}$, we have



and we can compute $g(1101) = 0011$ by following from g the path having input label 1101 and writing down the output.

The full graph for the odometer is of course infinite, but from any vertex, all long enough paths end in here:



Contracting groups

A self-similar group $G \leq \mathrm{Aut}(T_X)$ is called **contracting** if there exists a finite set $N \subseteq G$ such that

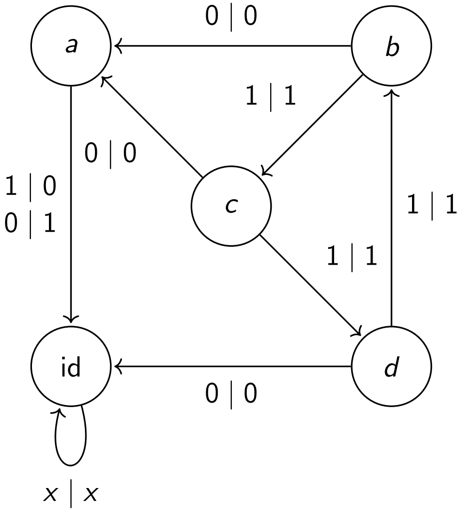
for all $g \in G$ there exists k with $g|_w \in N$ whenever $|w| \geq k$.

The minimal such N is called the **nucleus**.

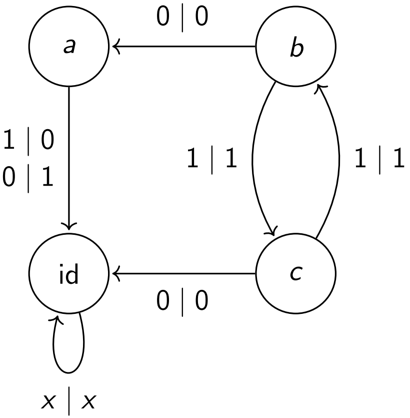
The odometer is contracting with nucleus $\{id, g, g^{-1}\}$.

If G is contracting, and generated by a finite set closed under sections, then there is an algorithm that computes N.

The generators (and nucleus) of the Grigorchuk group



The generators of the Grigorchuk-Erschler group



The nucleus is $\{a, b, c, bc, \mathrm{id}\}$.

Leavitt algebras

For $x \in X$ there is an operator ('push' x) $KX^\omega \to KX^\omega$ given by

$$x \cdot w = xw,$$

and an adjoint operator x^* ('pop' x) given by

$$x^* \cdot yw = \begin{cases} w & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

The operators satisfy the relations

- $x^* y = \delta_{x,y}$ (push y and pop x)
- $\sum_{x \in X} xx^* = 1$ (pop the first letter and push it back)

These operators generate the so-called Leavitt algebra $L_K(X)$. It can be defined abstractly using the above relations.

Self-similar groups and X^ω

If $g \in \mathrm{Aut}(T_X)$, the action of g on X^* extends to X^ω and KX^ω by

$$g(x_1x_2x_3\ldots) = g(x_1)g|_{x_1}(x_2)g|_{x_1x_2}(x_3)\ldots$$

The maps corresponding to g and $\{x, x^* : x \in X\}$ satisfy

- $gx = g(x)g|_x$,
- $x^*g = g|_{g^{-1}(x)}(g^{-1}(x))^*$.

The Nekrashevych algebra

G: self-similar group, X: set with $|X| > 2$.

The Nekrashevych algebra $N_K(G, X)$ is the K-algebra generated by the set G and $\{x, x^* : x \in X\}$, subject to the relations

- $g \cdot h = gh$,
- $gx = g(x)g|_x$,
- $x^*g = g|_{g^{-1}(x)}(g^{-1}(x))^*$,
- $x^*y = \delta_{x,y}$,
- $\sum_{x \in X} xx^* = 1$.

So $N_K(G, X)$ can be represented on KX^ω.

Nekrashevych first defined a C^*-algebra, and later studied this discrete counterpart.

Simplicity of Nekrashevych algebras

The question we were interested in is

when is $N_K(G, X)$ simple?

$N_{\mathbb{C}}(G, X)$ not simple $\Longrightarrow$ the C^*-algebra is not simple

- If G satisfies a condition called Hausdorff, then $N_K(G, X)$ is simple for any K
- Nekrashevych (2015): the Nekrashevych algebra of the Grigorchuk group is not simple over characteristic 2
- Clark, Exel, Pardo, Sims, Starling (2018): it is simple over all other characteristics
- Nekrashevych (2019): the Nekrashevych algebra of the Grigorchuk-Erschler group is simple over no field

The 'Hausdorff' condition

Take a self-similar group G,

- start with the state diagram for the group (if contracting, the nucleus suffices)
- delete all edges labeled by $x \mid y$ with $x \neq y$
- take the component of id
- G is Hausdorff if the obtained graph contains no cycles other than loops around id.

Main results

Theorem (Steinberg, Sz.)

$N_K(G, X)$ is simple if and only if its representation on KX^ω is faithful. Moreover, the image of the representation is the unique simple quotient of $N_K(G, X)$.

Theorem (Steinberg, Sz.)

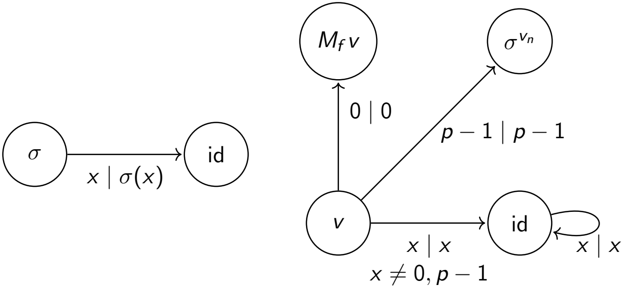
Let G be a contracting group with nucleus N.

- *Either $N_K(G, S)$ is simple over no field or simple over all but finitely many positive characteristics.*
- *There is an algorithm which on input N outputs the set of characteristics over which $N_K(G, S)$ is non-simple.*

For any finite set of primes $\mathcal{P}$, we give a contracting self-similar group G such that $N_K(G, X)$ fails to be simple exactly over characteristics in $\mathcal{P}$.

An example: Šunić groups

Fix a prime p and a non-constant polynomial $f \in \mathbb{Z}_p[x]$ with $f(0) \neq 0$, $n = \deg f$. Let M_f be the companion matrix of f.
Alphabet: $\mathbb{Z}_p$
Generators: $\{\sigma\} \cup \mathbb{Z}_p^n$, where σ acts on letters by the cyclic permutation $(0\ 1 \ldots p-1)$, and



The Grigorchuk group is a Šunić group with $p = 2$, $f = x^2 + x + 1$, the Grigorchuk-Erschler with $p = 2$, $f = x^2 + 1$.

Theorem (Steinberg, Sz.)

Let Šunić group $G_{p,f}$ be defined by p and $f \in \mathbb{Z}_p[x]$ with $\deg f = n$.

- *The Nekrashevych algebra is simple when $n = 1$;*
- *if $n > 1$ and M_f acts transitively on the 1-dimensional subspaces of $\mathbb{Z}_p^n$, then the algebra is simple over all fields except those with characteristic p;*
- *otherwise the Nekrashevych algebra is never simple.*