Inverse semigroup algebras and C*-algebras: the simplicity of Katsura algebras

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Definition

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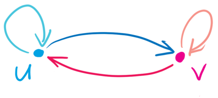
The canonical example

The set of partial one-to-one maps on a set A under composition and inverse: the symmetric inverse monoid $\mathrm{SIM}(A)$.

Graph inverse semigroups

Let E be a finite digraph with no sinks.



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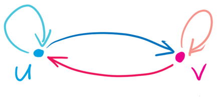

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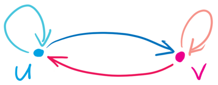

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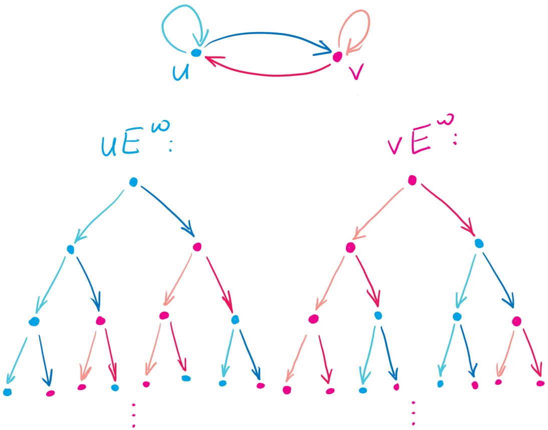

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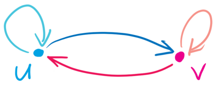

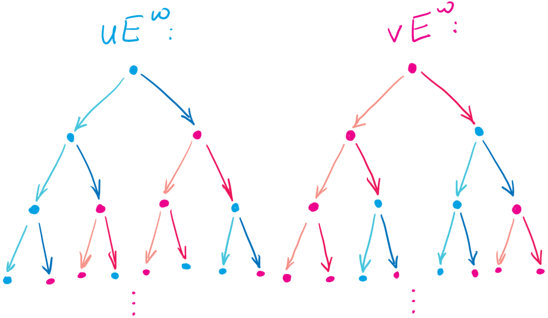

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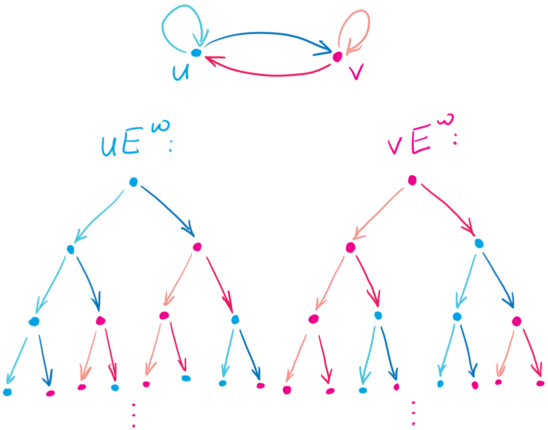

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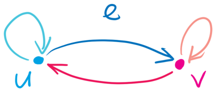

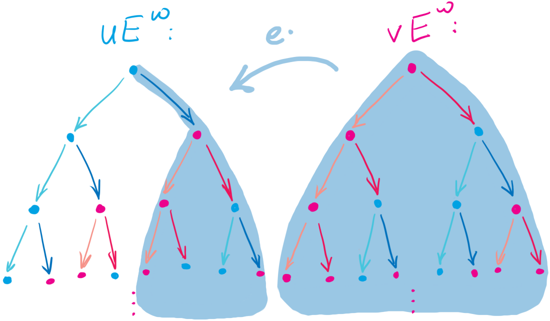

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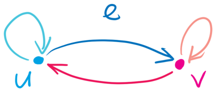

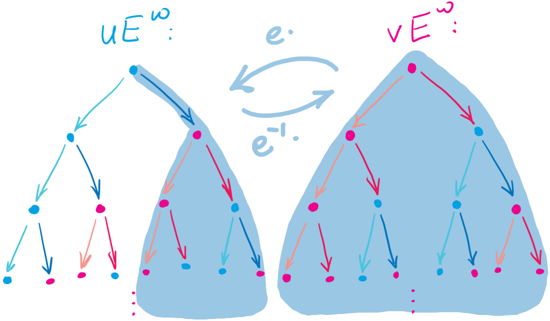

Graph inverse semigroups

Alternatively, S_E can be defined by the following inverse semigroup presentation:

Generators: $E^0 \cup E^1$

Relations:

(i) $uv = \begin{cases} u & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}$

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S_E has a normal form:

$$S_E = \{pq^{-1} \colon p, q \in E^*, r(q) = r(p)\} \cup \{0\}$$

Inverse semigroup algebras

Let S be a nontrivial inverse semigroup with a 0, and K a field.

The **contracted semigroup algebra** KS consists of finite linear combinations of nonzero elements of S over K. It is

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Note: A congruence $\sim$ on S defines a quotient $KS \to K(S/\sim)$, so if KS is simple then S is congruence-free.

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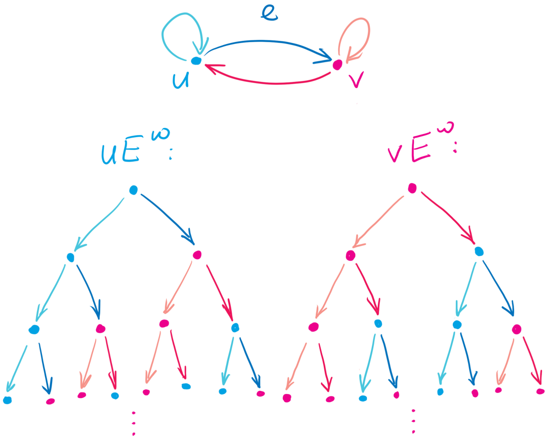

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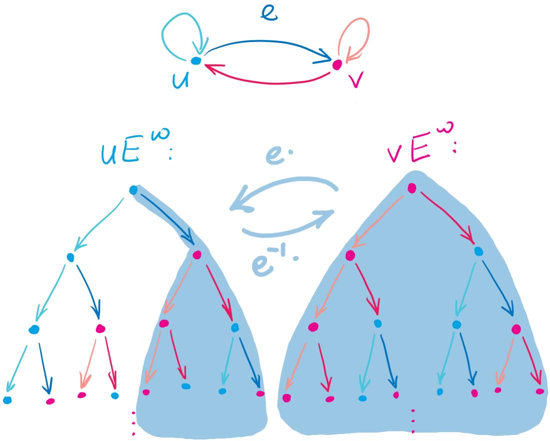

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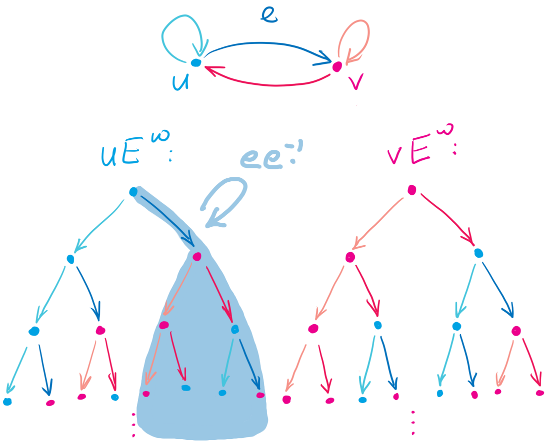

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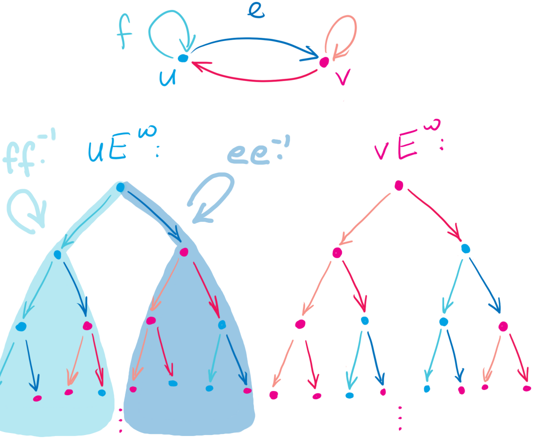

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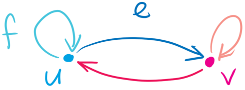

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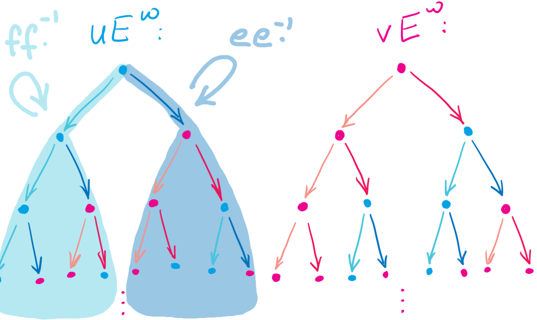

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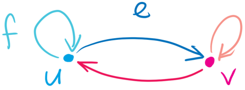

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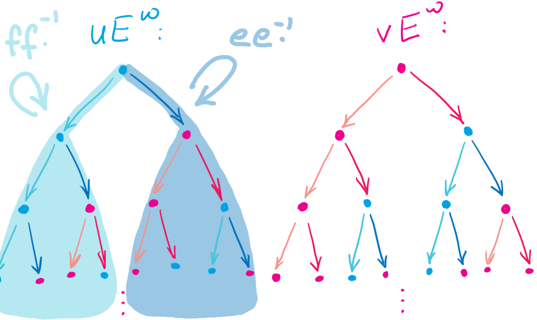
So the action of KS_E is non-faithful $\Longrightarrow$ KS_E is not simple.



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Fact: the simplicity of $L_K(E)$ for any field, and of $C^*(E)$ coincide and can be characterized in terms of the graph E. When S_E is congruence-free, $L_K(E)$ is simple.

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Theorem (Steinberg, Sz., 2021)

*Let S be a congruence-free inverse semigroup with 0. Then the **singular ideal** $I_K \lhd KS$ is the unique maximal ideal of KS. In particular, KS/I_K is simple.*

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Note: $T_K \triangleleft I_K$; and if S is congruence-free, then KS/T_K is simple iff $T_K = I_K$. In general, this is not easy to check.

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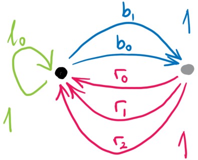
Remark: there is also an "unfaithful" Katsura algebra, which always quotients onto the faithful one, and is thus a less interesting candidate for simplicity.

Definition by example



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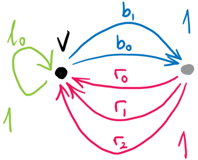
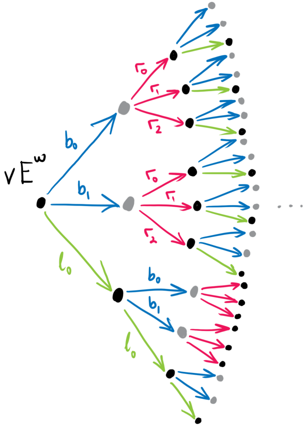
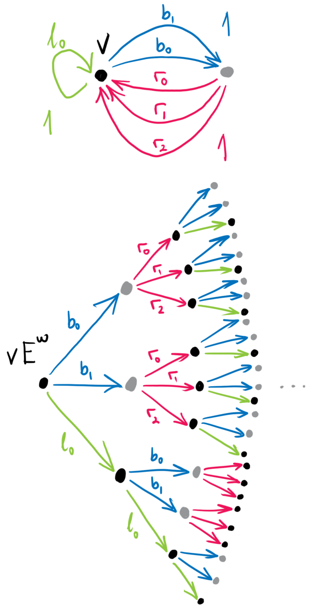


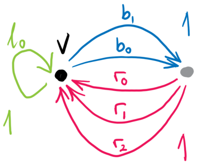
Let $v \in E^0$.
We define an action of $\mathbb{Z}$ on vE^ω, the action of 1 corresponding to a map g_v.

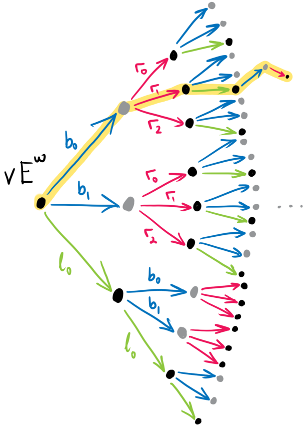




Let $v \in E^0$.
We define an action of $\mathbb{Z}$ on vE^ω, the action of 1 corresponding to a map g_v.
For each $p \in vE^\omega$, $n \in \mathbb{Z}$, $g_v^n(p) \in vE^\omega$ has the same vertex sequence (ie. coloring) as p.

Definition by example





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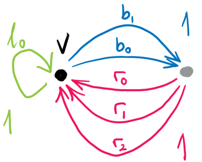

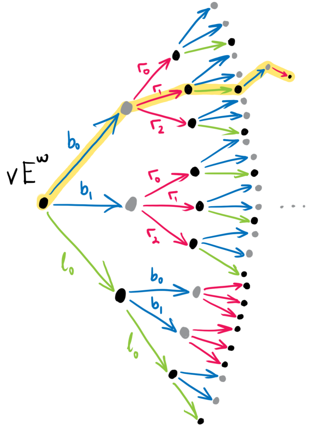

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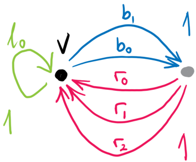
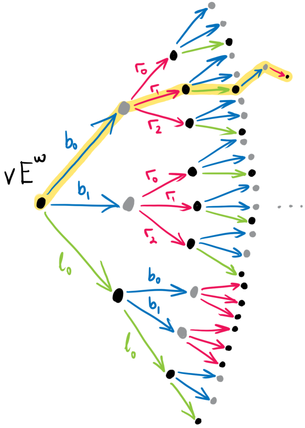
$$p := b_0 r_1 \ell_0 b_0 r_2 \ldots \leftrightarrow 01002 \ldots$$





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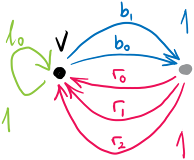
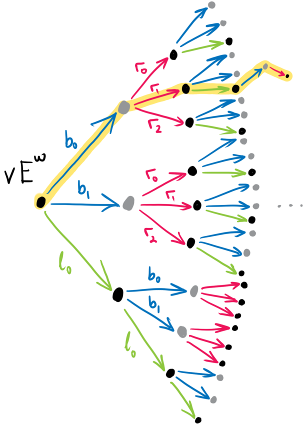
We display p as a sequence of numbers on a 'water meter' (first digit has least value):

$\boxed{0}$ $\boxed{1}$ $\boxed{0}$ $\boxed{0}$ $\boxed{2}$ ...

Definition by example





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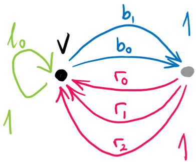
$$p := b_0 r_1 \ell_0 b_0 r_2 \ldots \leftrightarrow 01002\ldots$$

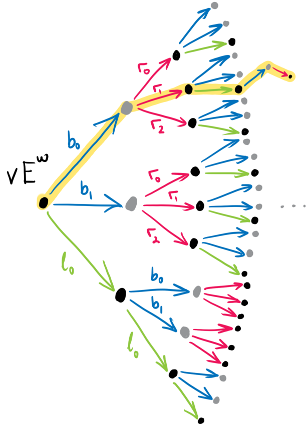
We display p as a sequence of numbers on a 'water meter' (first digit has least value):

0,1	0,1,2	0	0,1	0,1,2	
$\boxed{0}$	$\boxed{1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{2}$	$\ldots$

Each digit is on a dial, whose size is the number of edges of that color. When a dial completes a full rotation, the next dial is incremented.

Definition by example

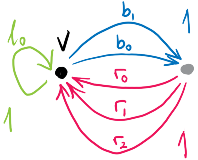


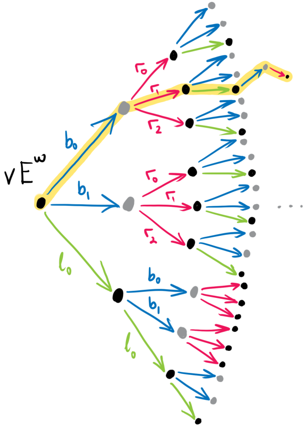


We calculate the path $g_v^n(p)$ by pumping n units of water through the meter.

$$g_v^9(b_0 r_1 \ell_0 b_0 r_2 \dots) =$$

$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{2} & \dots \end{array}$$

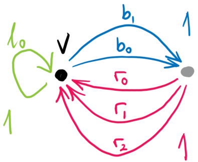


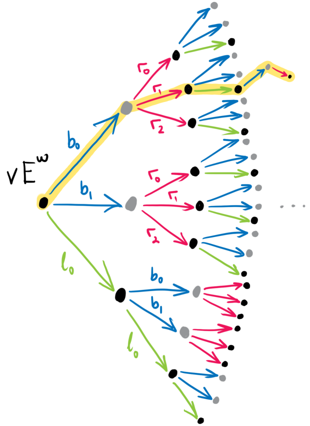


We calculate the path $g_v^n(p)$ by pumping n units of water through the meter.

$$g_v^9(b_0\, r_1\, \ell_0\, b_0\, r_2 \ldots) =$$

0,1	0,1,2	0	0,1	0,1,2	
$\boxed{0}$	$\boxed{1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{2}$	...
9					





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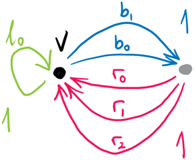
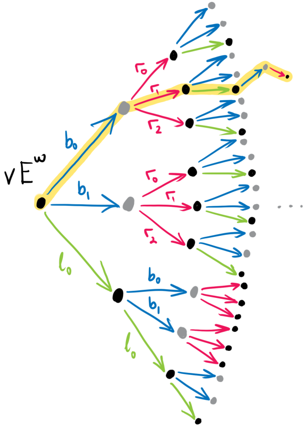
9					

$$9 + \boxed{0} = 4 \cdot 2 + \boxed{1}$$





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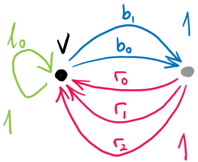
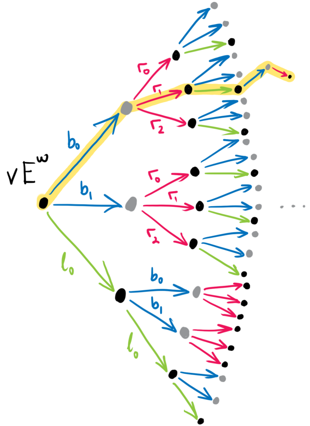
$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{1} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{2} & \ldots \\ & 4 & & & & \end{array}$$

$$9 + \boxed{0} = 4 \cdot 2 + \boxed{1}$$

Definition by example





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$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{1} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{2} & \dots \\ & 4 & & & & \end{array}$$

$$9 + \boxed{0} = 4 \cdot 2 + \boxed{1}$$
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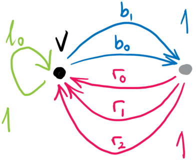
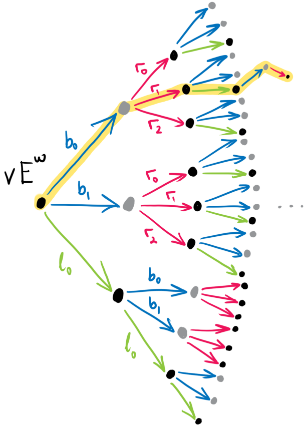
Definition by example





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0,1	0,1,2	0	0,1	0,1,2	
$\boxed{1}$	$\boxed{2}$	$\boxed{0}$	$\boxed{0}$	$\boxed{2}$	$\ldots$
		1			

$$9 + \boxed{0} = 4 \cdot 2 + \boxed{1}$$
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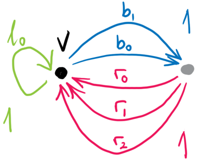
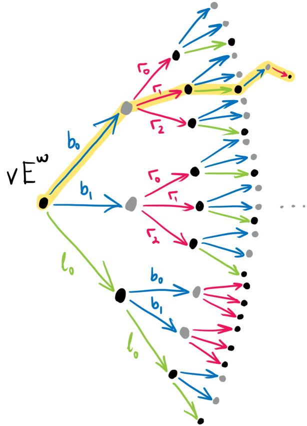
Definition by example





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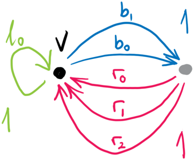
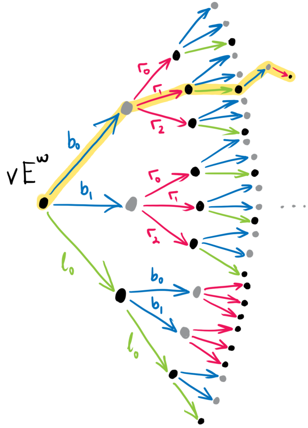
Definition by example





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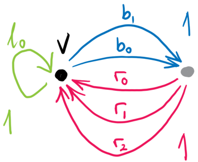

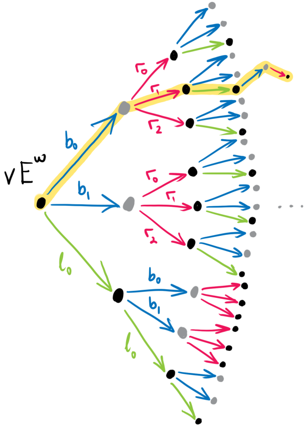

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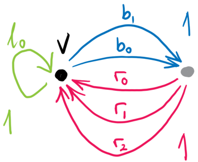

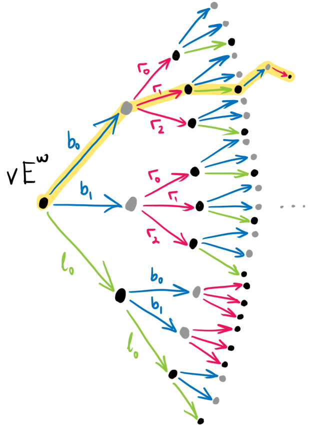

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$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{1} & \boxed{2} & \boxed{0} & \boxed{1} & \boxed{2} & \dots \end{array}$$

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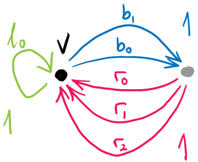
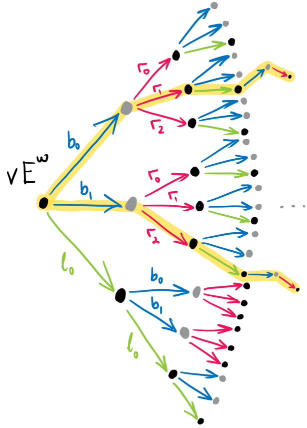
Definition by example





We calculate the path $g_v^n(p)$ by pumping n units of water through the meter.

$$g_v^9(b_0\, r_1\, \ell_0\, b_0\, r_2 \dots) = b_1\, r_2\, \ell_0\, b_1\, r_2 \dots$$

$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{1} & \boxed{2} & \boxed{0} & \boxed{1} & \boxed{2} & \dots \end{array}$$

$$9 + \boxed{0} = 4 \cdot 2 + \boxed{1}$$
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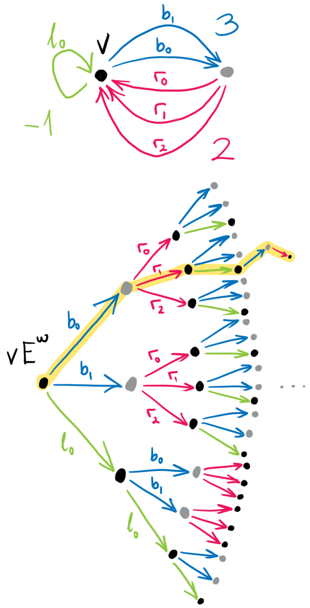

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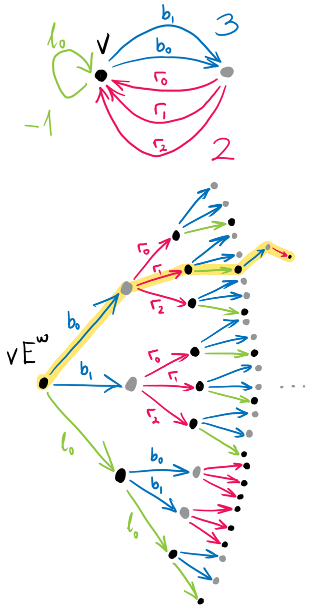

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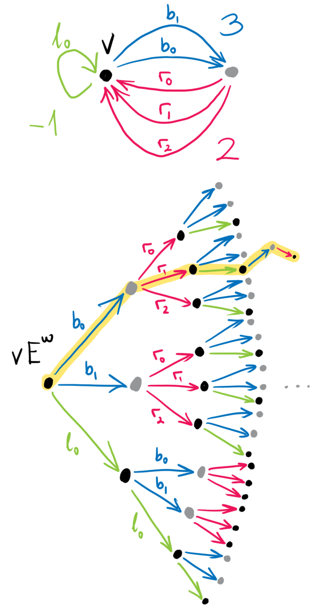
0,1	0,1,2	0	0,1	0,1,2	
$\boxed{0}$	$\boxed{1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{2}$	$\dots$
9					

Definition by example



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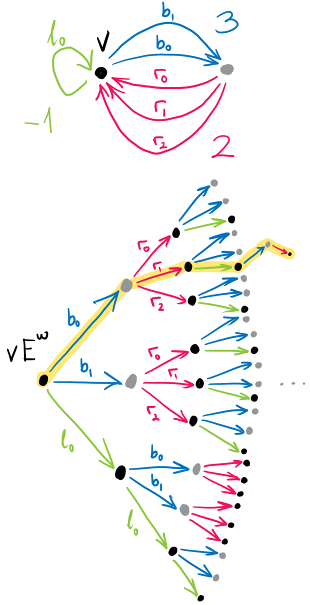
$$g_v^9(b_0 r_1 \ell_0 b_0 r_2 \ldots) =$$

$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{2} & \ldots \\ 9 & & & & & \end{array}$$

$$9 \cdot 3 + \boxed{0} = 13 \cdot 2 + \boxed{1}$$



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$$g_v^9(b_0 r_1 \ell_0 b_0 r_2 \ldots) =$$

$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{1} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{2} & \ldots \\ & 13 & & & & \end{array}$$

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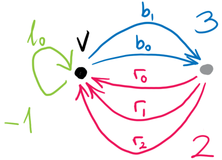
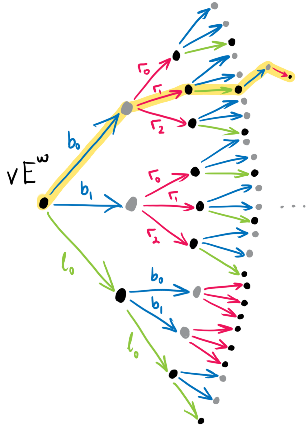
Definition by example





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0,1	0,1,2	0	0,1	0,1,2	
$\boxed{1}$	$\boxed{1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{2}$	$\ldots$
	13				

$$9 \cdot 3 + \boxed{0} = 13 \cdot 2 + \boxed{1}$$
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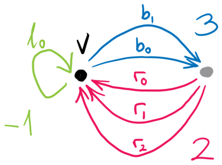
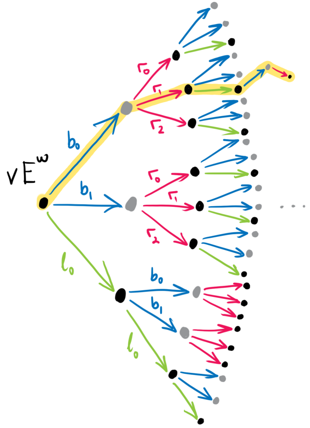
Definition by example





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$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{2} & \ldots \\ & & 9 & & & \end{array}$$

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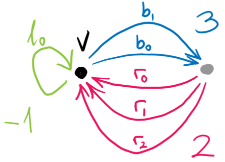
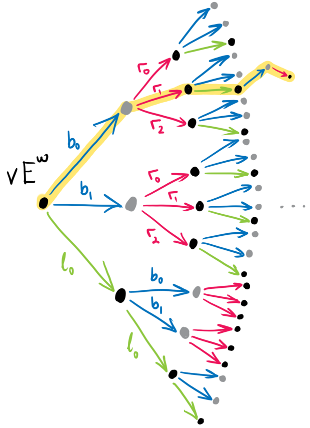
Definition by example





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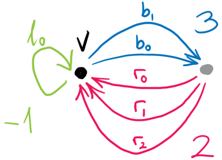
$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{2} & \ldots \\ & & 9 & & & \end{array}$$

$$9 \cdot 3 + \boxed{0} = 13 \cdot 2 + \boxed{1}$$
$$13 \cdot 2 + \boxed{1} = 9 \cdot 3 + \boxed{0}$$
$$9 \cdot (-1) + \boxed{0} = -9 \cdot 1 + \boxed{0}$$

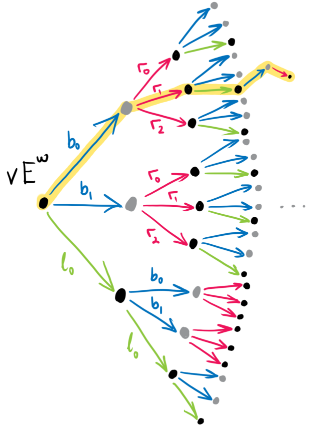
Definition by example



We calculate the path $g_v^n(p)$ by pumping n units of water through the meter.

$$g_v^9(b_0\, r_1\, \ell_0\, b_0\, r_2 \ldots) =$$

$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{2} & \ldots \\ & & & -9 & & \end{array}$$



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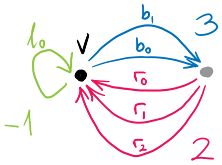
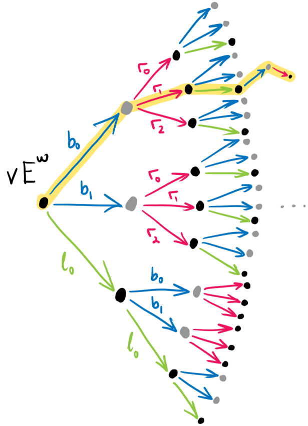




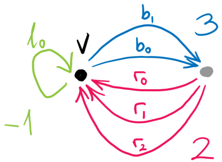
We calculate the path $g_v^n(p)$ by pumping n units of water through the meter.

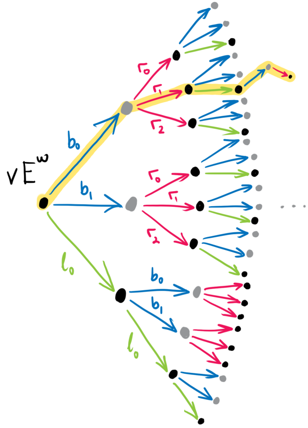
$$g_v^9(b_0 r_1 \ell_0 b_0 r_2 \dots) =$$

$$\overset{0,1}{\boxed{1}} \quad \overset{0,1,2}{\boxed{0}} \quad \overset{0}{\boxed{0}} \quad \underset{-\,9}{\overset{0,1}{\boxed{0}}} \quad \overset{0,1,2}{\boxed{2}} \quad \dots$$

$$9 \cdot 3 + \boxed{0} = 13 \cdot 2 + \boxed{1}$$
$$13 \cdot 2 + \boxed{1} = 9 \cdot 3 + \boxed{0}$$
$$9 \cdot (-1) + \boxed{0} = -9 \cdot 1 + \boxed{0}$$
$$-9 \cdot 3 + \boxed{0} = -14 \cdot 2 + \boxed{1}$$

Definition by example





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0,1	0,1,2	0	0,1	0,1,2	
$\boxed{1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{1}$	$\boxed{2}$	$\ldots$
				-14	

$$9 \cdot 3 + \boxed{0} = 13 \cdot 2 + \boxed{1}$$
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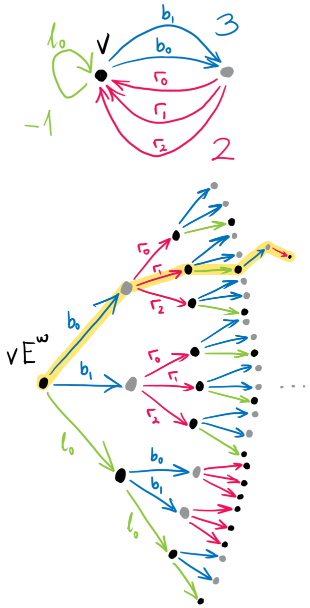
$$-9 \cdot 3 + \boxed{0} = -14 \cdot 2 + \boxed{1}$$

Definition by example



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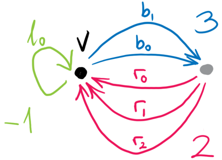
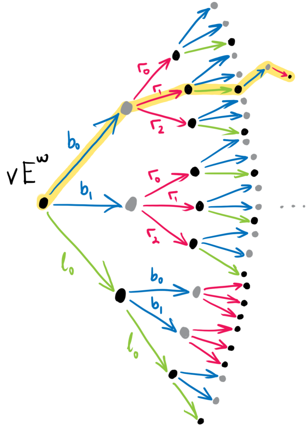
$$\overset{0,1}{\boxed{1}}\quad \overset{0,1,2}{\boxed{0}}\quad \overset{0}{\boxed{0}}\quad \overset{0,1}{\boxed{1}}\quad \underset{-14}{\overset{0,1,2}{\boxed{2}}}\quad \ldots$$

$$\begin{aligned}
9 \cdot 3 + \boxed{0} &= 13 \cdot 2 + \boxed{1} \\
13 \cdot 2 + \boxed{1} &= 9 \cdot 3 + \boxed{0} \\
9 \cdot (-1) + \boxed{0} &= -9 \cdot 1 + \boxed{0} \\
-9 \cdot 3 + \boxed{0} &= -14 \cdot 2 + \boxed{1} \\
-14 \cdot 2 + \boxed{2} &= -10 \cdot 3 + \boxed{0}
\end{aligned}$$

Definition by example





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0,1	0,1,2	0	0,1	0,1,2	
$\boxed{1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{1}$	$\boxed{0}$	$\dots$ -10

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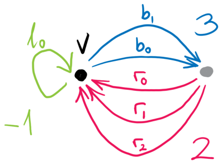

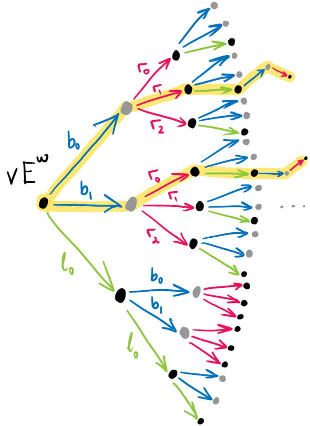

We calculate the path $g_v^n(p)$ by pumping n units of water through the meter.

$$g_v^9(b_0 r_1 \ell_0 b_0 r_2 \dots) = b_1 r_0 \ell_0 b_1 r_0 \dots$$

$$\overset{0,1}{\boxed{1}} \quad \overset{0,1,2}{\boxed{0}} \quad \overset{0}{\boxed{0}} \quad \overset{0,1}{\boxed{1}} \quad \overset{0,1,2}{\boxed{0}} \quad \dots$$

$$9 \cdot 3 + \boxed{0} = 13 \cdot 2 + \boxed{1}$$
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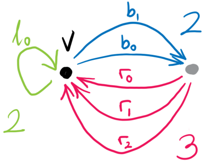
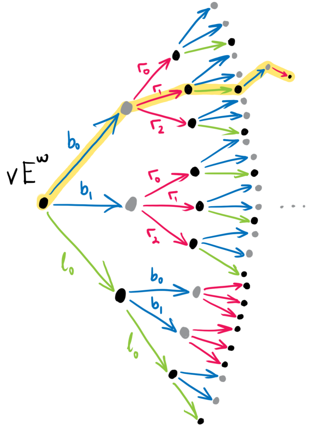


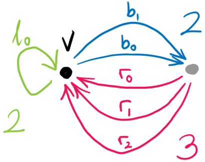


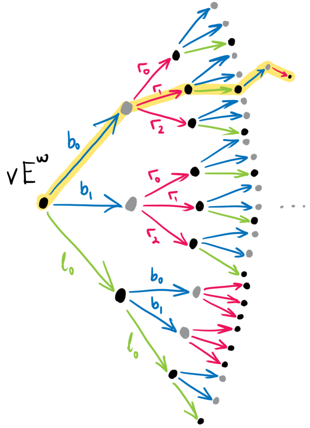
We calculate the path $g_v^n(p)$ by pumping n units of water through the meter.

$$g_v^9(b_0\, r_1\, \ell_0\, b_0\, r_2 \ldots) =$$

0,1	0,1,2	0	0,1	0,1,2	
$\boxed{0}$	$\boxed{1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{2}$	...

One more example





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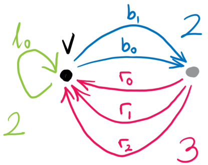
0,1	0,1,2	0	0,1	0,1,2	
$\boxed{0}$	$\boxed{1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{2}$	...
9					

One more example



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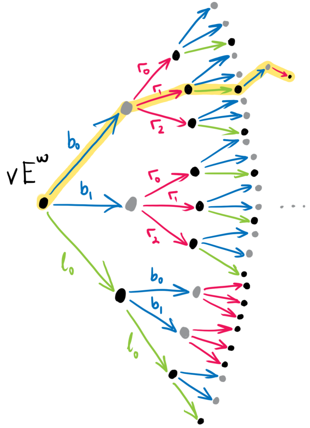
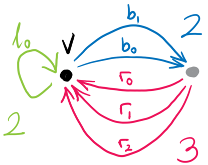
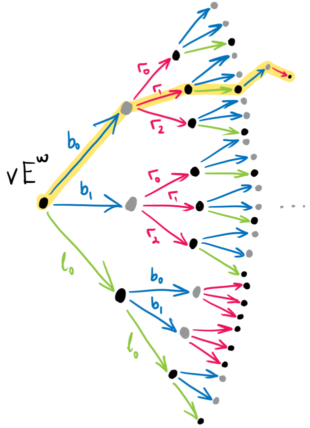


$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{2} & \dots \\ 9 & & & & & \end{array}$$

$$9 \cdot 2 + \boxed{0} = 9 \cdot 2 + \boxed{0}$$

One more example





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0,1	0,1,2	0	0,1	0,1,2	
$\boxed{0}$	$\boxed{1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{2}$	...
	9				

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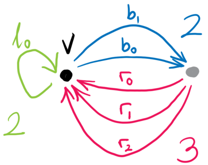

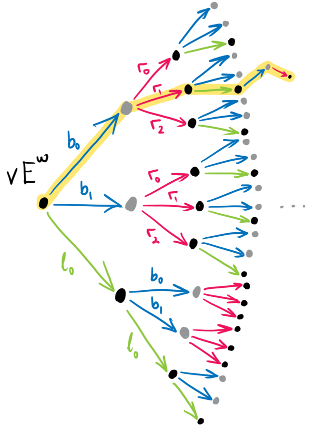

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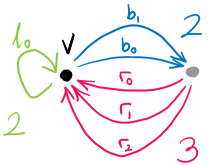
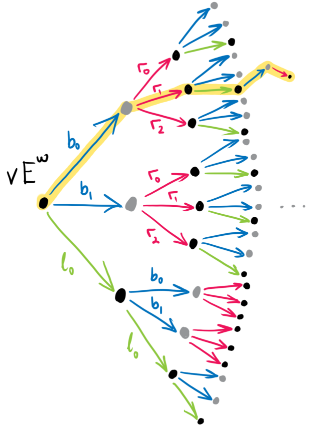
$$9 \cdot 2 + \boxed{0} = 9 \cdot 2 + \boxed{0}$$
$$9 \cdot 3 + \boxed{1} = 9 \cdot 3 + \boxed{1}$$

One more example





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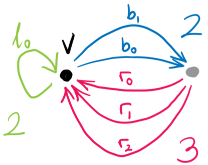
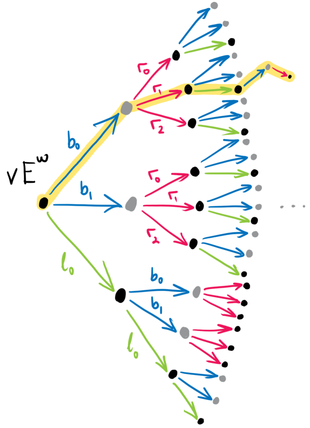
$$9 \cdot 3 + \boxed{1} = 9 \cdot 3 + \boxed{1}$$

One more example





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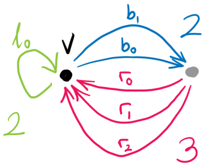
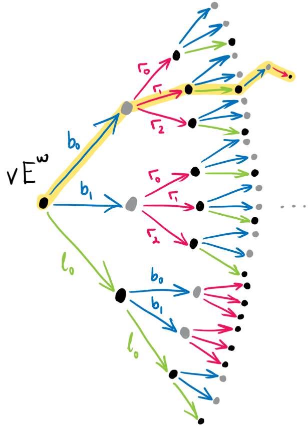
$$9 \cdot 2 + \boxed{0} = 9 \cdot 2 + \boxed{0}$$
$$9 \cdot 3 + \boxed{1} = 9 \cdot 3 + \boxed{1}$$
$$9 \cdot 2 + \boxed{0} = 18 \cdot 1 + \boxed{0}$$

One more example





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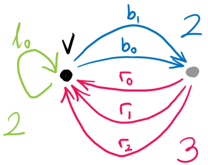
$$\overset{0,1}{\boxed{0}} \quad \overset{0,1,2}{\boxed{1}} \quad \overset{0}{\boxed{0}} \quad \underset{18}{\overset{0,1}{\boxed{0}}} \quad \overset{0,1,2}{\boxed{2}} \quad \dots$$

$$9 \cdot 2 + \boxed{0} = 9 \cdot 2 + \boxed{0}$$
$$9 \cdot 3 + \boxed{1} = 9 \cdot 3 + \boxed{1}$$
$$9 \cdot 2 + \boxed{0} = 18 \cdot 1 + \boxed{0}$$



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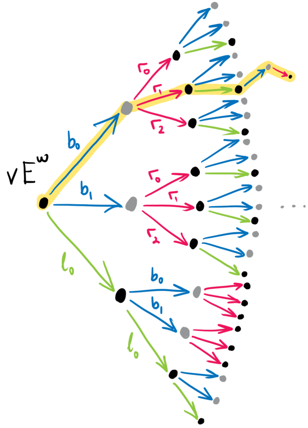
$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{2} & \dots \\ & & & 18 & & \end{array}$$

$$9 \cdot 2 + \boxed{0} = 9 \cdot 2 + \boxed{0}$$
$$9 \cdot 3 + \boxed{1} = 9 \cdot 3 + \boxed{1}$$
$$9 \cdot 2 + \boxed{0} = 18 \cdot 1 + \boxed{0}$$
$$18 \cdot 2 + \boxed{0} = 18 \cdot 2 + \boxed{0}$$



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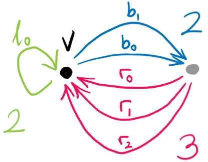

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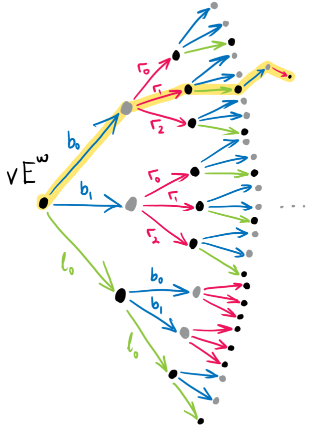
$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{2} & \ldots \\ & & & & 18 & \end{array}$$



$$9 \cdot 2 + \boxed{0} = 9 \cdot 2 + \boxed{0}$$
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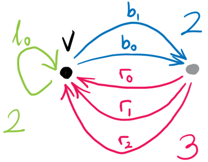
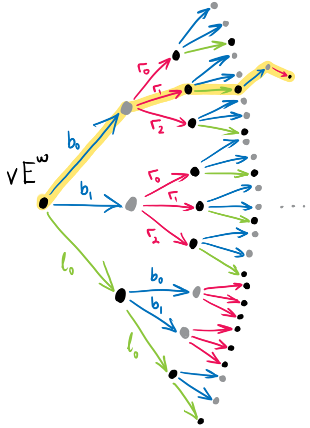
One more example





We calculate the path $g_v^n(p)$ by pumping n units of water through the meter.

$$g_v^9(b_0\, r_1\, \ell_0\, b_0\, r_2 \ldots) = b_0\, r_1\, \ell_0\, b_0\, r_2 \ldots = p$$

$$\begin{array}{cccccc} 0,1 & 0,1,2 & 0 & 0,1 & 0,1,2 & \\ \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{2} & \ldots \\ & & & & 18 & \end{array}$$

$$9 \cdot 2 + \boxed{0} = 9 \cdot 2 + \boxed{0}$$
$$9 \cdot 3 + \boxed{1} = 9 \cdot 3 + \boxed{1}$$
$$9 \cdot 2 + \boxed{0} = 18 \cdot 1 + \boxed{0}$$
$$18 \cdot 2 + \boxed{0} = 18 \cdot 2 + \boxed{0}$$

The inverse semigroup $S_{E,B}$

Recall: $S_E = \{pq^{-1} \colon p, q \in E^*, r(q) = r(p)\} \cup \{0\} \leq \mathrm{SIM}(E^\omega)$

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$$\text{Define } S_{E,B} := \langle g_v \colon v \in E^0, S_E \rangle \leq \mathrm{SIM}(E^\omega).$$

Note: g_v can have finite order.

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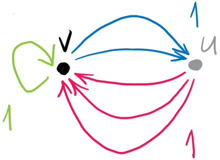
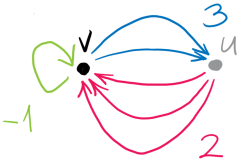
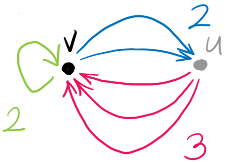


$o(g_v) = o(g_u) = \infty$



$o(g_v) = 2, o(g_u) = 3$



$o(g_v) = o(g_u) = 1$

The inverse semigroup $S_{E,B}$

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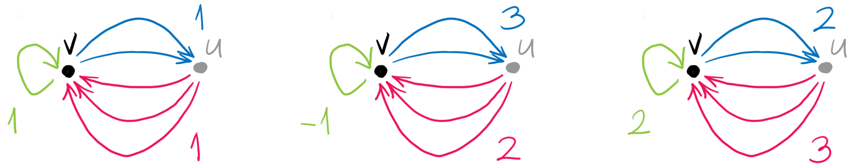
$$\text{Define } S_{E,B} := \langle g_v : v \in E^0, S_E \rangle \leq \mathrm{SIM}(E^\omega).$$

Note: g_v can have finite order.

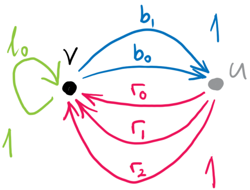


$o(g_v) = o(g_u) = \infty$ $\qquad$ $o(g_v) = 2, o(g_u) = 3$ $\qquad$ $o(g_v) = o(g_u) = 1$

Remark: the order of g_v is computable in polynomial space on input E and B. [Aakre, Sz., 2026+]

Self-similarity

Recall our previous computation:



$$g_v^9(b_0\, r_1\, \ell_0\, b_0\, r_2 \ldots) =$$

0,1	0,1,2	0	0,1	0,1,2	
$\boxed{0}$	$\boxed{1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{2}$	...

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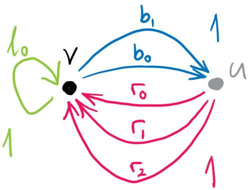

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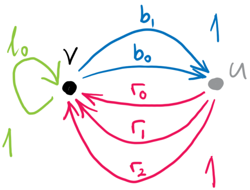

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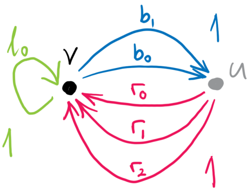

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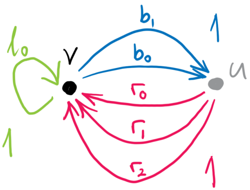

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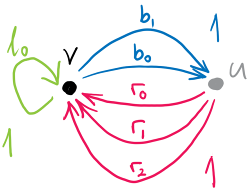

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So

$$g_v^9 b_0 = b_1 g_u^4 \text{ in } S_{E,B}.$$

Normal form

More generally, if the $u \to v$ edges are $e_0, \ldots e_{A(u,v)-1}$, then

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In particular, $S_{E,B}$ is finitely presented by the relations:

- defining S_E
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If we further restrict n between 0 and $o(g_v)$ whenever the order is finite, then the above is a normal form.

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Remark: we prove similar results for the 'unfaithful' Katsura algebras; these are much easier to obtain.