The large scale geometry of inverse semigroups

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A crash course on geometric group theory

A key idea: derive algebraic and algorithmic properties of f.g. groups from the (large scale) geometry of their Cayley graphs.

For example: hyperbolic groups have a decidable word problem *(Gromov, 1987)*

In the subsequent years, Gromov outlined a long-term plan to study finitely generated groups via their (large scale) geometry, which led to the development of geometric group theory as a stand alone field.

To what extent can this be generalized to larger classes of semigroups?

Inverse semigroups

A semigroup S is called an inverse semigroup if every element $s \in S$ has a unique inverse s^{-1} satisfying

$$ss^{-1}s = s, s^{-1}ss^{-1} = s^{-1}.$$

The Cayley graph of an inverse semigroup is much less nice than a group Cayley graph, in particular:

- ss^{-1} is typically not 1, so typically edges labeled by s and s^{-1} don't come in reverse pairs
- the graph is typically not strongly connected (e.g. a 0 element is a sink)

This makes it difficult to define a natural metric on the Cayley graph that is related to the algebraic properties of the inverse semigroup.

Schützenberger graphs

However, the strong components of the Cayley graph are nice.

The vertices s and t are in the same strong component iff there are directed paths from s to t and t to s.

Equivalently, if $s \mathcal{R} t$, that is, $ss^{-1} = tt^{-1}$.

Fact: if $s \mathcal{R} sx$, then $sxx^{-1} = s \implies$ within the strong components, edges **do** come in inverse pairs.

The strong components are called Schützenberger graphs, and they are naturally metric spaces with the path metric.

Finitely generated inverse semigroups as metric spaces

$\mathcal{S}_X$: the graph formed as the disjoint union of all Schützenberger graphs of $S = \langle X \rangle$.

$\mathcal{S}_X$ still retains enough information about S:

- S can be reconstructed from $\mathcal{S}_X$,
- the word problem boils down to deciding the languages of *automata* with components of $\mathcal{S}_X$ as underlying graphs: *Schützenberger automata*

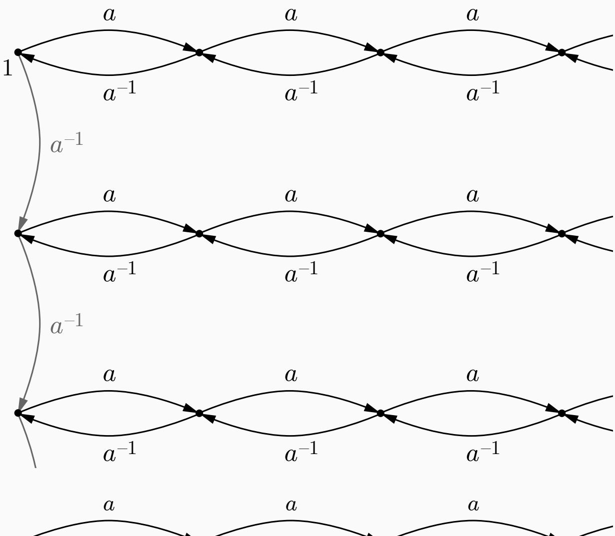
So we can define the word metric metric on S as:

$d_X(s,t) =$ the path distance of s and t in $\mathcal{S}_X$

$(d_X(s,t) = \infty$ when s and t are in different components)

Example: the bicyclic semigroup

$S = Inv\langle a \mid aa^{-1} = 1\rangle$



Geometry and the word problem

A natural question: Do all finite presented, hyperbolic inverse semigroups have a solvable word problem?

No. *(Gray, Silva, Sz., 2021)*

In a finitely presented inverse semigroup where all Schützenberger graphs are quasi-isometric to trees, the word problem is solvable (and the languages of the Schützenberger automata are context-free). *(Gray, Silva, Sz., 2021)*

Geometry and C*-algebras

A key application of inverse semigroups: using inverse semigroups to construct (and study) C*-algebras.

A C*-algebra is a subset $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ of bounded linear operators on a complex Hilbert space (think infinite matrices!) which is closed under linear combinations, products, adjoints, and closed in operator norm.

Given a (adjectives omitted) metric space X (such as that coming from a graph via the path metric), there is an associated C*-algebra called the uniform Roe algebra of the space: $C^*_u(X) \subseteq \mathcal{B}(\ell^2(X))$.

It encapsulates the large scale properties of the space:

$$C^*_u(X) \cong C^*_u(Y) \iff X \text{ and } Y \text{ are bijectively large scale equivalent}$$

Uniform Roe algebras of groups

If G is a group equipped with a metric, then

$$C_u^*(G) \cong \ell^\infty(G) \rtimes_r G,$$

where the latter C*-algebra is obtained directly from the Cayley representation of G.

If S is an inverse semigroup, we can similarly construct $\ell^\infty(S) \rtimes_r S$ from the Wagner-Preston representation.

Natural question: is this also a uniform Roe algebra of some metric on S?

If S is f.g. and equipped with the word metric, then $C_u^*(S) \cong \ell^\infty(S) \rtimes_r S$. *(Lledó, Martínez, 2021)*

Quasi-countable inverse semigroups

This generalizes to non-finitely generated inverse semigroups.

We call S is quasi-countable if it's generated by $X \cup E(S)$ for some countable set X.

Any quasi-countable inverse semigroup can be equipped with a (adjectives ommitted) metric in a large scale unique way, and for this (any such) metric $C_u^*(S) \cong \ell^\infty(S) \rtimes_r S$. *(Chung, Martínez, Sz., 2022)*

Finiteness properties

We investigate the following finiteness properties of C*-algebras:

(1) local AF $\implies$ (2) strongly quasi-diagonal $\implies$ (3) quasi-diagonal $\implies$ (4) stably finite $\implies$ (5) finite

Given a quasi-countable inverse semigroup S, we characterize in terms of S, and in terms of the metric, when $C_u^*(S)$ satisfies the above conditions *(Chung, Martínez, Sz., 2022)*

(1) $\iff$ (2) $\iff$ S is locally finite $\iff$ S has asymtotic dimension 0

(3) $\iff$ (4) $\iff$ (5) $\iff$ S is locally $\mathcal{R}$-finite $\iff$ S is sparse