Closure operators on group Cayley graphs, and presentations of F-inverse monoids

Panglobal Algebra and Logic Seminar

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An introduction to inverse monoids

Inverse monoids

Definition
An inverse semigroup is a semigroup M such that for all $a \in M$, there is a unique $a^{-1} \in M$ such that $a = aa^{-1}a$, $a^{-1} = a^{-1}aa^{-1}$.

Inverse semigroups form a variety with the operations $(M, \cdot, ^{-1})$, defined by associativity and the identities

$$a = aa^{-1}a, \; a^{-1} = a^{-1}aa^{-1}, \; aa^{-1}bb^{-1} = bb^{-1}aa^{-1}.$$

Additional nice identities: $(a^{-1})^{-1} = a$, $(ab)^{-1} = b^{-1}a^{-1}$

We can similarly define inverse monoids.

Note: every element of an X-generated inverse monoid can be represented by the term in $(X \cup X^{-1})^*$.

Free inverse monoids

Munn gave the following, beautiful description of the X-generated free inverse monoid $\mathrm{FIM}(X)$:

- elements: (T, g), where $g \in \mathrm{FG}(X)$, and T is a finite subtree of the Cayley graph of $\mathrm{FG}(X)$ with $1, g \in \Delta$
- product: $(T, g)(U, h) = (T \cup gU, gh)$
- inverse: $(T, g) = (g^{-1}T, g^{-1})$

Margolis-Meakin expansions

Note: we can replace $\mathsf{FG}(X)$ in the previous construction with the Cayley graph Γ_X of any X-generated group G, and consider the inverse monoid $M(G)$ with

- elements: (Δ, g), where $g \in G$, and Δ is a finite subgraph of Γ_X with $1, g \in T$
- product: $(\Delta, g)(\Xi, h) = (\Delta \cup g\Xi, gh)$
- inverse: $(\Delta, g)^{-1} = (g^{-1}\Delta, g^{-1})$

This is called the Margolis-Meakin expansion of G, and it is the initial object in the category of X-generated [additional adjectives] inverse monoids.

Additional adjectives

Any inverse monoid M has a least group congruence σ: the one generated by the pairs $(xx^{-1}, 1)$. M/σ is called the greatest group image.

M is called E-unitary if 1σ is exactly the set of idempotents of M.

Definition
Given an X-generated group G, consider the category $\mathcal{E}(G, X)$ with

- objects: X-generated, E-unitary inverse monoids with greatest group image G
- morphisms: inverse semigroup morphisms fixing the generating set X

$M(G)$ is the initial object of $\mathcal{E}(G, X)$.

A structure theorem for E-unitary inverse monoids

Theorem (McAlister's P-theorem)
Every E-unitary inverse monoid can be built (in a given way) out of

- *a group G (the greatest group image),*
- *a semilattice $\mathcal{Y}$ (the semilattice of idempotents),*
- *a poset $\mathcal{X}$ containing $\mathcal{Y}$, acted upon by G.*

Steinberg gave a proof of McAlister's theorem where $\mathcal{X}$ is given as a certain set of connected subgraphs of the Cayley graph of G, and $\mathcal{Y}$ consists of those graphs in $\mathcal{X}$ which contain 1.

We rephrase this approach in terms of closure operators on the subgraphs of the Cayley graph of G.

Closure operators and E-unitary inverse semigroups

Closure operators

- G: an X-generated group
- Γ_X: its Cayley graph
- $\mathrm{CSub}\,\Gamma_X$: the set of connected subgraphs of Γ_X
- $()^c$: a closure operator on the poset $(\mathrm{CSub}\,\Gamma_X, \subseteq)$

$()^c$ is G-invariant if the set of closed graphs is invariant under the left action of G.

$()^c$ is finitary/algebraic if $\Delta^c = \bigcup\{F^c : F \subseteq \Delta \text{ is finite}\}$.

$\Delta \in \mathrm{CSub}\,\Gamma_X$ is compact if it is the closure of a finite subgraph.

Inverse semigroups from closure operators

Given a G-invariant, finitary closure operator on $(\mathrm{CSub}\,\Gamma_X, \subseteq)$, we can define an inverse monoid M_c:

- elements: (Δ, g), where $g \in G$, and Δ is a compact subgraph of Γ_X with $1, g \in \Delta$
- product: $(\Delta, g)(\Xi, h) = ((\Delta \cup g\Xi)^c, gh)$
- inverse: $(\Delta, g)^{-1} = (g^{-1}\Delta, g^{-1})$

M^c is also E-unitary and X-generated.

Closure operators

Definition
Given an X-generated group G, consider the category $\mathcal{C}(G, X)$ with

- objects: finitary, G-invariant closure operators on $(\mathrm{CSub}\,\Gamma_X, \subseteq)$
- morphisms: there is a morphism from $()^{c_1}$ to $()^{c_2}$ iff all c_2-closed subgraphs are c_1-closed

Recall: $\mathcal{E}(G, X)$ is the category of X-generated, E-unitary inverse monoids with greatest group image G

Theorem
The categories $\mathcal{E}(G, X)$ and $\mathcal{C}(G, X)$ are equivalent.

F-inverse monoids

F-inverse monoids

Inverse monoids are naturally equipped with a partial order:
$s \leq t \iff s = ts^{-1}s$

An inverse monoid M is F-inverse if every σ-class has a greatest element.

The variety of F-inverse monoids

F-inverse monoids form a variety with the operations $(\cdot, 1, ^{-1}, ^{\mathfrak{m}})$ where $a^{\mathfrak{m}}$ is the greatest element is the σ-class of a. The defining identites are those of inverse monoids and

$$a^{\mathfrak{m}} \geq a, \; a^{\mathfrak{m}} = (abb^{-1})^{\mathfrak{m}}.$$

Additional nice identities: $(ab^{\mathfrak{m}}c)^{\mathfrak{m}} = (abc)^{\mathfrak{m}}, (a^{\mathfrak{m}})^{-1} = (a^{-1})^{\mathfrak{m}}$

As such, every element of an X-generated F-inverse monoid can be represented by the term

$$u_1 v_1^{\mathfrak{m}} u_2 v_2^{\mathfrak{m}} \cdots v_{n-1}^{\mathfrak{m}} u_n,$$

$u_i, v_i \in (X \cup X^{-1})^*$.

Free F-inverse monoids

Auinger, Kudryavtseva and M. Szendrei gave the following description of the X-generated free inverse F-inverse monoid:

- elements: (T, g), where $g \in \mathrm{FG}(X)$, and T is a finite subgraph (not necessarily connected!) of the Cayley graph of $\mathrm{FG}(X)$ with $1, g \in \Delta$
- product: $(T, g)(U, h) = (T \cup gU, gh)$
- inverse: $(T, g)^{-1} = (g^{-1}T, g^{-1})$
- maximum element in σ-class: $(T, g)^{\mathfrak{m}} = (\{1, g\}, g)$

They also define the analogues of Margolis-Meakin expansions, and prove that they are the initial objects in the category of X-generated F-inverse monoids with greatest group image G.

Closure operators again

- G: an X-generated group
- Γ_X: its Cayley graph
- $\operatorname{Sub}\Gamma_X$: the set of all subgraphs of Γ_X
- $()^c$: a closure operator on the poset $(\operatorname{Sub}\Gamma_X, \subseteq)$

Given a G-invariant, finitary closure operator on $(\operatorname{Sub}\Gamma_X, \subseteq)$, we can define an X-generated F-inverse monoid M^c:

- elements: (Δ, g), where $g \in G$, and Δ is a compact subgraph of Γ_X with $1, g \in \Delta$
- product: $(\Delta, g)(\Xi, h) = ((\Delta \cup g\Xi)^c, gh)$
- inverse: $(\Delta, g)^{-1} = (g^{-1}\Delta, g^{-1})$
- maximum element in σ-class: $(\Delta, g)^{\mathfrak{m}} = (\{1, g\}^c, g)$

Closure operators from F-inverse monoids

To every X-generated F-inverse monoid M with $M/\sigma = G$, we associate a finitary, G-invariant closure operator $()^{c_M}$ on $\operatorname{Sub}\Gamma_X$. We define $\Delta \in \operatorname{Sub}\Gamma_X$ to be closed iff

for any two terms w_1, w_2 with $w_1 \equiv_M w_2$, and for any $g \in G$, Δ contains the journey from g labeled by w_1 if and only if it contains the journey from g labeled by w_2.

Proposition
If M is given by a presentation (in the variety of F-inverse monoids), it suffices to consider pairs of relators as w_1, w_2 in the condition above.

The equivalence of categories

$\mathcal{F}(X, G)$: the category of X-generated, F-inverse monoids with greatest group image G

$\mathcal{S}(X, G)$: the category of closure operators on $\operatorname{Sub}\Gamma_X$

Theorem
The categories $\mathcal{F}(G, X)$ and $\mathcal{S}(G, X)$ are equivalent.

Presentations of F-inverse monoids

Let $M = \mathrm{FInv}\langle X \mid R\rangle$ be a presentation of F-inverse monoid, and let $G = M/\sigma$.

Define the F-Schützenberger graph $F\Gamma(w)$ of a term w to be the smallest subgraph of Γ_X which

- contains the journey labeled by w starting at 1;
- is closed in c_M.

Theorem

1. *Given two term w_1, w_2, we have $w_1 \geq_M w_2$ iff $w_1 \equiv_G w_2$ and $F\Gamma(w_1) \subseteq F\Gamma(w_2)$.*
2. *In particular, $w_1 =_M w_2$ iff $w_1 \equiv_G w_2$ and $F\Gamma(w_1) = F\Gamma(w_2)$.*

Example: the free inverse monoid

Recall the free inverse monoid $\mathsf{FIM}(X)$.

This is given by the F-inverse presentation $\mathsf{FInv}\langle X \mid w^{\mathfrak{m}} = \mathsf{red}(w)\rangle$.