Bimonoidal adjunctions

Paul Taylor
Honorary Research Fellow
University of Birmingham

PSSL 101, Leeds
Saturday 16 September 2017

`www.Paul Taylor.EU/slides/17-PSSL-Leeds.pdf`

Web address for notes available on request.

Funded by: my late parents.

Both ends of an adjunction are equally important

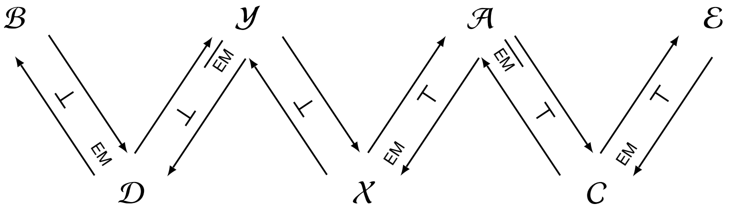
Applications:

- descent: somebody please explain this to me;
- Hopf algebras: ditto;
- quantum computation: (I know whom to ask about this);
- localic locales or colocales: Steve Vickers (and me);
- linear–non-linear models: Nick Benton & Gavin Bierman (plus ideas of mine);
- Stone duality in general: what makes the opposite of a category of algebras suitable to be a kind of "category of spaces"?
- building new interesting categories

There are various papers about these applications, but they have fragmentary bits of theory.

Relevant pure theory is in very sophisticated settings (closed, enriched or higher dimensional categories) without a good overall explanation.

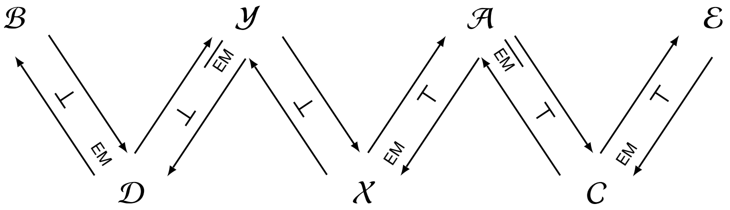
Iterating Eilenberg–Moore and its dual



Adjunction $\mathcal{Y} \leftrightarrows \mathcal{X}$ gives monad on $\mathcal{X}$ gives adjunction $\mathcal{A} \leftrightarrows \mathcal{X}$ gives comonad on $\mathcal{A}$ gives adjunction $\mathcal{A} \leftrightarrows \mathcal{C}$ gives monad on $\mathcal{C}$ gives ...

Does this stabilise?

Iterating Eilenberg–Moore and its dual

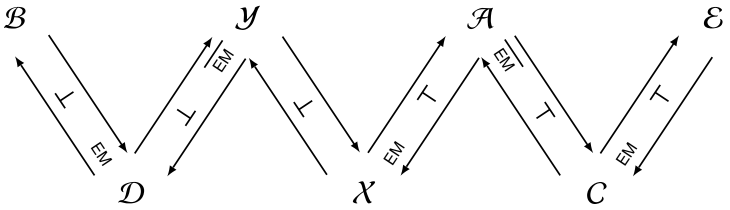


Adjunction $\mathcal{Y} \leftrightarrows \mathcal{X}$ gives monad on $\mathcal{X}$ gives adjunction $\mathcal{A} \leftrightarrows \mathcal{X}$ gives comonad on $\mathcal{A}$ gives adjunction $\mathcal{A} \leftrightarrows \mathcal{C}$ gives monad on $\mathcal{C}$ gives ...

Does this stabilise?

Yes, Steve Lack sent me the proof in a fax in 1999, but apparently this is not well known.

Iterating Eilenberg–Moore and its dual



Adjunction $\mathcal{Y} \leftrightarrows \mathcal{X}$ gives monad on $\mathcal{X}$ gives adjunction $\mathcal{A} \leftrightarrows \mathcal{X}$ gives comonad on $\mathcal{A}$ gives adjunction $\mathcal{A} \leftrightarrows \mathcal{C}$ gives monad on $\mathcal{C}$ gives ...

Does this stabilise?

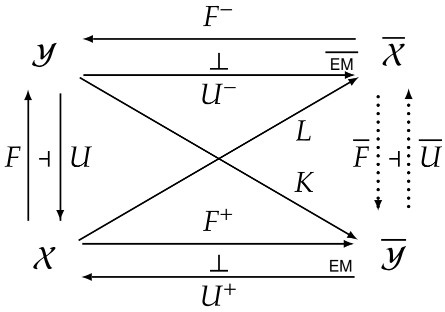
Yes, Steve Lack sent me the proof in a fax in 1999, but apparently this is not well known.

Does it give the same result the other way?

Adjunction $\mathcal{Y} \leftrightarrows \mathcal{X}$ gives comonad on $\mathcal{Y}$ gives adjunction $\mathcal{Y} \leftrightarrows \mathcal{D}$ gives monad on $\mathcal{D}$ gives adjunction $\mathcal{B} \leftrightarrows \mathcal{D}$ gives comonad on $\mathcal{B}$ gives ...

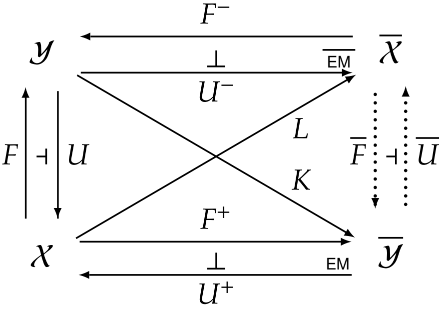
Eilenberg–Moore commutes with its dual

A neater proof, starting with the adjunction not a monad.
Form the Eilenberg–Moore categories at each end separately:



Eilenberg–Moore commutes with its dual

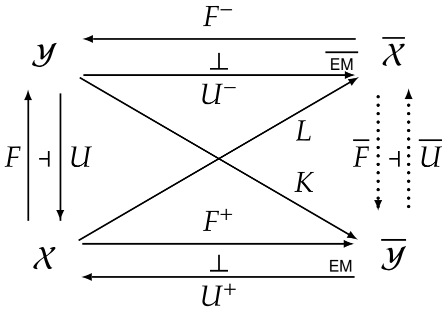
A neater proof, starting with the adjunction not a monad.
Form the Eilenberg–Moore categories at each end separately:



There's a new adjunction $\overline{F} \dashv \overline{U}$.
Is it monadic and comonadic?

Eilenberg–Moore commutes with its dual

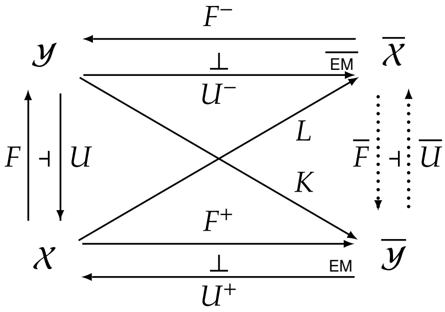
A neater proof, starting with the adjunction not a monad.
Form the Eilenberg–Moore categories at each end separately:



There's a new adjunction $\overline{F} \dashv \overline{U}$.
Is it monadic and comonadic? Not in general.
But it's premonadic and precomonadic,
i.e. the comparison functors are full and faithful.

Eilenberg–Moore commutes with its dual

A neater proof, starting with the adjunction not a monad.
Form the Eilenberg–Moore categories at each end separately:

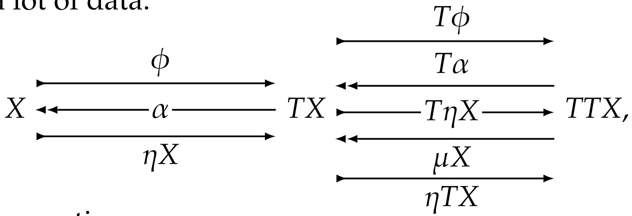


There's a new adjunction $\overline{F} \dashv \overline{U}$.
Is it monadic and comonadic? Not in general.
But it's premonadic and precomonadic,
i.e. the comparison functors are full and faithful.
It's monadic and comonadic if idempotents split.

Coalgebras for a monad

I'm not sure what to call these,
but this name should make people stop and think.

Rather a lot of data:



with the equations

$$\eta X\,;T\eta X \;=\; \eta X \;=\; \eta TX \qquad \alpha\,;\eta X \;=\; \eta TX\,;T\alpha$$

$$\eta X\,;\alpha \;=\; \mathsf{id}_X \;=\; \phi\,;\alpha \qquad T\alpha\,;\alpha \;=\; \mu X\,;\alpha$$

$$\alpha\,;\phi \;=\; T\phi\,;\mu X \qquad \phi\,;T\phi \;=\; \phi\,;T\eta X \;=\; \eta TX.$$

It would be nice to cut down on the data,
but it can come in handy...

Ordered monads

A KZ-monad on an ordered category satisfies

$$T\eta X \;\leq\; \eta T X.$$

Main property:
$\alpha : TX \to X$ is an algebra structure map iff $\alpha \;\dashv\; \eta X$

Such monads add certain joins to order structures (possibly with existing joins or meets).

Ordered monads

A KZ-monad on an ordered category satisfies

$$T\eta X \;\leq\; \eta TX.$$

Main property:
$\alpha : TX \to X$ is an algebra structure map iff $\alpha \dashv \eta X$
Such monads add certain joins to order structures
(possibly with existing joins or meets).

Extension to coalgebras:
$\phi : X \to TX$ is a coalgebra structure map iff $\phi \dashv \alpha \dashv \eta X$

Continuous algebras

$$\begin{array}{c} \mathbf{CSLat} \\ F \left\uparrow \dashv \right\downarrow U \\ \mathbf{SLat} \end{array} \qquad \begin{array}{c} \mathbf{CSLat} \\ F \left\uparrow \dashv \right\downarrow U \\ \mathbf{Pos} \end{array}$$

FX is the lattice of ideals or lower sets.

$$\alpha_X U \equiv \bigvee U \quad \text{and} \quad \eta_X x \equiv \downarrow x$$

Continuous algebras

$$\begin{array}{c} \mathbf{CSLat} \\ F \Big\uparrow \dashv \Big\downarrow U \\ \mathbf{SLat} \end{array} \qquad \begin{array}{c} \mathbf{CSLat} \\ F \Big\uparrow \dashv \Big\downarrow U \\ \mathbf{Pos} \end{array}$$

FX is the lattice of ideals or lower sets.

$\alpha_X U \equiv \bigvee U$ and $\eta_X x \equiv \downarrow x$

$\phi_X x \equiv \twoheaddownarrow x$ way below

There are similar results for continuous lattices
and Richard Wood's CCD lattices.
I'm not sure who deserves credit for this: maybe
Peter Johnstone and André Joyal 1982
Barry Fawcett and Richard Wood 1990 or Bart Jacobs 1994.
No similar characterisation for co-KZ-monads adding meets.

Limits and colimits are hopeless!

Limits of algebras are easy, but we lose colimits.
Colimits of coalgebras are easy, but we lose limits.
So for coalgebras on algebras, we've lost everything!

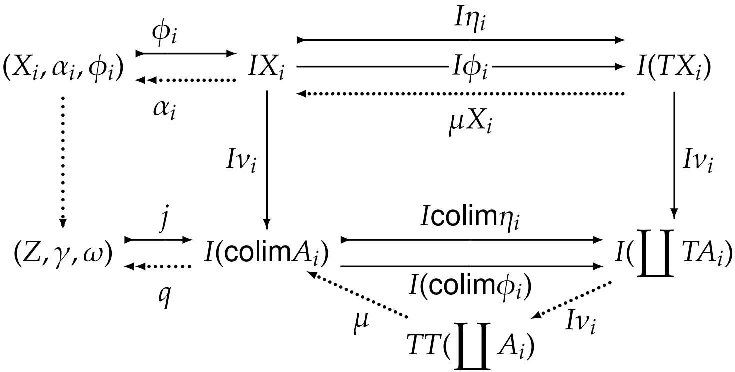
Limits and colimits are hopeless!

Limits of algebras are easy, but we lose colimits.
Colimits of coalgebras are easy, but we lose limits.
So for coalgebras on algebras, we've lost everything!

Not so!
We take advantage of all the extra arrows in the structure...

Colimits are easy!

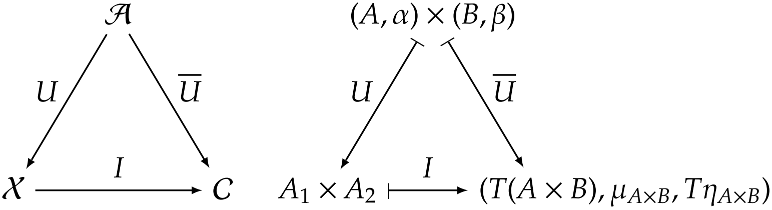
If idempotents split in $\mathcal{X}$ then coalgebras admit whatever colimits the base category had and the comparison functor preserves them.



This adapts Fred Linton's result for colimits of algebras, but without assuming that the monad preserves reflexive coequalisers.

Products of free coalgebras

If the objects $A, B \in \mathcal{X}$ carry algebra structures (say, α and β, though they will play no role) then the product $IA \star IB$ exists in $\mathcal{C}$, and is given by $I(A \times B)$, with projections $I\pi_0$ and $I\pi_1$.



The functors U and $\overline{U}$ are right adjoints and so preserve limits, in particular binary products.

Every coalgebra is an equaliser of free ones, so we should be able to extend this result to all coalgebras.

Give me strength!

It is very unlikely that categories of coalgebras for a monad have general equalisers.
(For example, colocales are frames with structure, their equalisers are subframes or quotient locales, but there may be a proper class of these, so we cannot use the Special Adjoint Functor Theorem.)

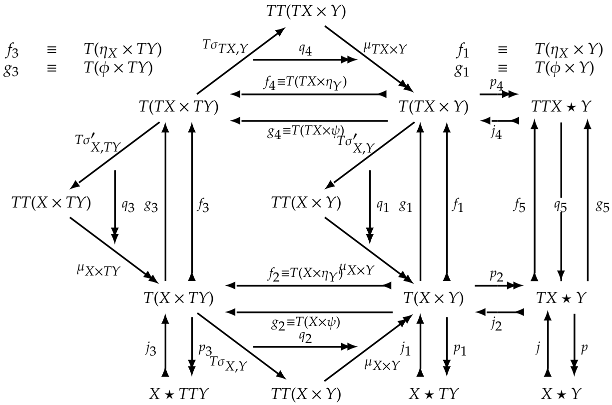
So we're looking for split equalisers.

These require extra structure, provided by a strength.

$$\sigma_{X,Y} : X \times TX \longrightarrow T(X \times Y).$$

This does not need to be "commutative".

Products of coalgebras



The f and g squares $T(X \times Y) \rightrightarrows T(TX \times TY)$ commute.
The q squares the other way need not commute, but either composite provides the required splitting.

The same diagram without the coalgebras

We try to define a coequaliser of algebras:

$$
\begin{array}{ccccc}
F(TA \times TB) & \overset{q_4}{\underset{h_4}{\rightrightarrows}} & F(TA \times B) & \xrightarrow{p_4} & FTA \otimes (B,\beta) \\
{\scriptstyle q_3}\downarrow\downarrow{\scriptstyle h_3} & & {\scriptstyle q_1}\downarrow\downarrow{\scriptstyle h_1} & & {\scriptstyle q_5}\downarrow\downarrow{\scriptstyle h_5} \\
F(A \times TB) & \overset{q_2}{\underset{h_2}{\rightrightarrows}} & F(A \times B) & \xrightarrow{p_2} & FA \otimes (B,\beta) \\
{\scriptstyle p_3}\downarrow & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p_5}\downarrow \\
(A,\alpha) \otimes FTB & \overset{q_6}{\underset{h_6}{\rightrightarrows}} & (A,\alpha) \otimes FB & \xrightarrow{p_6} & (A,\alpha) \otimes (B,\beta)
\end{array}
$$

For the q-square, σ must be commutative.
This is a general reflexive coequaliser, not a split one.

Commutative and monoidal monads

The strength σ is commutative if

$$\sigma_{TX,Y} \,;\, T\sigma'_{X,Y} \;=\; \sigma'_{X,TY} \,;\, T\sigma_{X,Y} : TX \times TX \longrightarrow T(X \times Y)$$

where this composite is called $\kappa_{X,Y}$.

The axioms for κ make (T, η, μ, κ) a monoidal monad.

Commutative and monoidal are equivalent,
but the diagrams are horrendous!

Yet another equivalent form:

$$\sigma_{X,Y} \;=\; \eta_{X \times TY} \,;\, (TX \otimes \mu_Y).$$

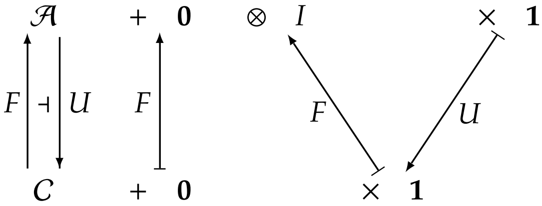
Theorem: any commutative or monoidal monad that preserves reflexive coequalisers defines tensor products of algebras and

$$F\mathbf{1} \cong I \qquad F(X \times Y) \cong FX \otimes FY.$$

Linear–non-linear models in Logic

Nick Benton and Gavin Bierman.

Jean-Yves Girard's Linear Logic with of course !



Eugenio Moggi's Computational Monads

Linear–non-linear models in Mathematics

Algebras

$$\mathbf{VSp} \underset{U}{\overset{F}{\leftrightarrows}} \mathbf{Set} \quad (F \dashv U) \qquad \mathbf{SLat} \underset{U}{\overset{F}{\leftrightarrows}} \mathbf{Set} \quad (F \dashv U) \qquad \mathbf{CSLat} \underset{U}{\overset{F}{\leftrightarrows}} \mathbf{Set} \quad (F \dashv U)$$

Carriers (all comonadic)

Algebras

$$\mathbf{CSLat} \underset{U}{\overset{F}{\leftrightarrows}} \mathbf{Pos} \quad (F \dashv U) \qquad \mathbf{CSLat} \underset{U}{\overset{F}{\leftrightarrows}} \mathbf{SLat} \quad (F \dashv U) \qquad \mathbf{CSLat} \underset{U}{\overset{F}{\leftrightarrows}} \mathbf{Dcpo} \quad (F \dashv U) \qquad \mathbf{PreFrm} \underset{U}{\overset{F}{\leftrightarrows}} \mathbf{Dcpo} \quad (F \dashv U)$$

Carriers (the coalegebras are the continuous ones)

Linear–non-linear models in Mathematics

Internal algebras with other carriers
(generalised "spaces")

$$
\begin{array}{c}
\mathbf{AbTopGp} \\
F \uparrow \dashv \downarrow U \\
\mathbf{Sp}
\end{array}
\qquad
\begin{array}{c}
\mathbf{AbAlgGp} \\
F \uparrow \dashv \downarrow U \\
\mathbf{AffVar}
\end{array}
\qquad
\begin{array}{c}
\mathbf{AbLocGp} \\
F \uparrow \dashv \downarrow U \\
\mathbf{Loc}
\end{array}
\qquad
\begin{array}{c}
\mathbf{SLatLoc} \\
F \uparrow \dashv \downarrow U \\
\mathbf{Loc}
\end{array}
$$

Linear–non-linear models in Mathematics

Internal algebras with other carriers
(generalised "spaces")

$$
\begin{array}{cccc}
\mathbf{AbTopGp} & \mathbf{AbAlgGp} & \mathbf{AbLocGp} & \mathbf{SLatLoc} \\
F \uparrow \dashv \downarrow U & F \uparrow \dashv \downarrow U & F \uparrow \dashv \downarrow U & F \uparrow \dashv \downarrow U \\
\mathbf{Sp} & \mathbf{AffVar} & \mathbf{Loc} & \mathbf{Loc}
\end{array}
$$

The underlying algebraic structure:
(we need the EM-completions of these)

$$
\begin{array}{ccc}
\mathbf{AbGp}^{\mathsf{op}} & \mathbf{CSLat}^{\mathsf{op}} & \mathbf{PreFrm}^{\mathsf{op}} \\
F \uparrow \dashv \downarrow U & F \uparrow \dashv \downarrow U & F \uparrow \dashv \downarrow U \\
\mathbf{CRing}^{\mathsf{op}} & \mathbf{Frm}^{\mathsf{op}} & \mathbf{Frm}^{\mathsf{op}}
\end{array}
$$

Stone duality

Familiar picture with multiplication over addition:

$$
\begin{array}{ccccc}
\mathbf{CRing} & \qquad & \mathbf{Frm} & \qquad & \mathbf{Frm} \\
F \Big\uparrow \dashv \Big\downarrow U & & F \Big\uparrow \dashv \Big\downarrow U & & F \Big\uparrow \dashv \Big\downarrow U \\
\mathbf{AbGp} & & \mathbf{CSLat} & & \mathbf{PreFrm} \\
F \Big\uparrow \dashv \Big\downarrow U & & F \Big\uparrow \dashv \Big\downarrow U & & F \Big\uparrow \dashv \Big\downarrow U \\
\mathbf{Set} & & \mathbf{Dcpo} & & \mathbf{Dcpo}
\end{array}
$$

Stone duality

Turning it upside down:

$$
\begin{array}{ccc}
\mathbf{Set}^{\mathrm{op}} & \mathbf{Dcpo}^{\mathrm{op}} & \mathbf{Dcpo}^{\mathrm{op}} \\
F \uparrow \dashv \downarrow U & F \uparrow \dashv \downarrow U & F \uparrow \dashv \downarrow U \\
\mathbf{AbGp} & \mathbf{CSLat} & \mathbf{PreFrm} \\
F \uparrow \dashv \downarrow U & F \uparrow \dashv \downarrow U & F \uparrow \dashv \downarrow U \\
\mathbf{CRing}^{\mathrm{op}} & \mathbf{Frm}^{\mathrm{op}} & \mathbf{Frm}^{\mathrm{op}}
\end{array}
$$

$\mathbf{Set}^{\mathrm{op}} \leftrightarrows \mathbf{CRing}^{\mathrm{op}}$ is already bi-monadic.
We replace $\mathbf{Dcpo}^{\mathrm{op}}$ by the EM-completion,
which we call localic frames
and its opposite localic locales or colocales.
CSLat and **PreFrm** also need to be EM-completed.

Stone duality

Turning it upside down again:

$$
\begin{array}{ccc}
\mathbf{Frm} & \qquad & \mathbf{CoLoc}^{\mathsf{op}} \\
F \big\uparrow \dashv \big\downarrow U & & F \big\uparrow \dashv \big\downarrow U \\
\mathcal{A} & & \mathcal{A}^{\mathsf{op}} \\
F \big\uparrow \dashv \big\downarrow U & & F \big\uparrow \dashv \big\downarrow U \\
\mathbf{CoLoc} & & \mathbf{Loc}
\end{array}
$$

where $\mathcal{A}$ is the EM-completion of either **CSLat** or **PreFrm**.

Composing monads

Jon Beck's distributive law $\delta : ST \to TS$.

Algebras for the composite monad (*e.g.* rings or frames)
are equivalent to
algebras for each monad (sum/join and product/meet)
compatible with the distributive law δ.

Composing monads

Jon Beck's distributive law $\delta : ST \to TS$.

Algebras for the composite monad (*e.g.* rings or frames)
are equivalent to
algebras for each monad (sum/join and product/meet)
compatible with the distributive law δ.

Coalgebras for the composite monad (*e.g.* rings or frames)
are equivalent to
coalgebras for each monad
compatible with the inverse δ^{-1}.

Composing monads

Jon Beck's distributive law $\delta : ST \to TS$.

Algebras for the composite monad (*e.g.* rings or frames)
are equivalent to
algebras for each monad (sum/join and product/meet)
compatible with the distributive law δ.

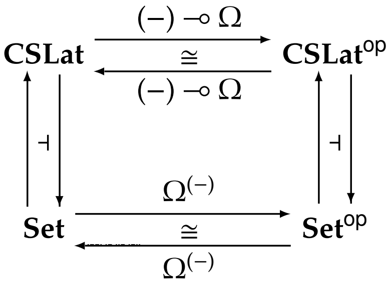
Coalgebras for the composite monad (*e.g.* rings or frames)
are equivalent to
coalgebras for each monad
compatible with the inverse δ^{-1}.

The powerlocales commute, *i.e.* the distributive law has an inverse.

Richard Wood: upper and lower subsets have distribitivities but they're adjoint, not inverse.

Linear–non-linear models again

The algebras may have a dualising object:



Set $\to$ **CSLat** is the covariant powerset.
Set $\to$ **Set**^{op} is the contravariant powerset.

Linear–non-linear models again

The algebras may have a dualising object:

$$\begin{array}{ccc} \mathbf{CSLat} & \underset{(-)\multimap\Omega}{\overset{(-)\multimap\Omega}{\rightleftarrows}}\ \cong & \mathbf{CSLat}^{\mathsf{op}} \\ \uparrow \dashv \downarrow & & \uparrow \dashv \downarrow \\ \mathbf{Pos} & \underset{\Omega^{(-)}}{\overset{\Omega^{(-)}}{\rightleftarrows}}\ \cong & \mathbf{CCD}^{\mathsf{op}} \end{array}$$

What about the two powerlocales?

Maybe we have to use both.
Self-duality would swap them round, like Chu.
Monadic Chu for the spaces (as in my talk in Cork).
Fertile ground for new interesting categories.
Finitary Properties of categories of “spaces”
without using anything like set theory.
Therefore potentially with a recursive version.