Stone duality for infinitary first-order logic

Christian Espíndola

PSSL 101

September 16th, 2017

Reconstruction results

Usual reconstruction results allow to get the syntactic category of the theory $\mathbf{T}$, possibly up to some form of completion, by identifying some structure in the category of models $Mod(\mathbf{T})$ and considering the category of functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving that structure.

Reconstruction results

Usual reconstruction results allow to get the syntactic category of the theory $\mathbf{T}$, possibly up to some form of completion, by identifying some structure in the category of models $Mod(\mathbf{T})$ and considering the category of functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving that structure.
For finitary theories the following results are known:

Reconstruction results

Usual reconstruction results allow to get the syntactic category of the theory $\mathbf{T}$, possibly up to some form of completion, by identifying some structure in the category of models $Mod(\mathbf{T})$ and considering the category of functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving that structure.

For finitary theories the following results are known:

- (Lawvere) Algebraic theories: Determined up to Cauchy completion by the functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits, filtered colimits and regular epimorphisms.

Reconstruction results

Usual reconstruction results allow to get the syntactic category of the theory $\mathbf{T}$, possibly up to some form of completion, by identifying some structure in the category of models $Mod(\mathbf{T})$ and considering the category of functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving that structure.

For finitary theories the following results are known:

- (Lawvere) Algebraic theories: Determined up to Cauchy completion by the functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits, filtered colimits and regular epimorphisms.
- (Gabriel-Ulmer) Cartesian theories: Determined up to equivalence by functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits and filtered colimits.

Reconstruction results

Usual reconstruction results allow to get the syntactic category of the theory $\mathbf{T}$, possibly up to some form of completion, by identifying some structure in the category of models $Mod(\mathbf{T})$ and considering the category of functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving that structure.
For finitary theories the following results are known:

- (Lawvere) Algebraic theories: Determined up to Cauchy completion by the functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits, filtered colimits and regular epimorphisms.
- (Gabriel-Ulmer) Cartesian theories: Determined up to equivalence by functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits and filtered colimits.
- (Makkai) Regular theories: Determined up to exact completion by functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving products and filtered colimits.

Reconstruction results

Usual reconstruction results allow to get the syntactic category of the theory $\mathbf{T}$, possibly up to some form of completion, by identifying some structure in the category of models $Mod(\mathbf{T})$ and considering the category of functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving that structure.
For finitary theories the following results are known:

- (Lawvere) Algebraic theories: Determined up to Cauchy completion by the functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits, filtered colimits and regular epimorphisms.
- (Gabriel-Ulmer) Cartesian theories: Determined up to equivalence by functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits and filtered colimits.
- (Makkai) Regular theories: Determined up to exact completion by functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving products and filtered colimits.
- (Makkai) Coherent theories: Determined up to pretopos completion by functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving ultraproducts and ultramorphisms (ultrafunctors) with certain natural transformations that are compatible with the ultraproducts (ultratransformations).

Reconstruction results

For infinitary theories in $\mathcal{L}_{\kappa,\kappa}$ for regular κ we have the following generalizations:

Reconstruction results

For infinitary theories in $\mathcal{L}_{\kappa,\kappa}$ for regular κ we have the following generalizations:

- (Adamek-Koubek-Velebil) κ-algebraic theories: Determined up to Cauchy completion by the functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits, κ-filtered colimits and regular epimorphisms.

Reconstruction results

For infinitary theories in $\mathcal{L}_{\kappa,\kappa}$ for regular κ we have the following generalizations:

- (Adamek-Koubek-Velebil) κ-algebraic theories: Determined up to Cauchy completion by the functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits, κ-filtered colimits and regular epimorphisms.
- (Makkai-Pitts) κ-cartesian theories: Determined up to equivalence by functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits and κ-filtered colimits.

Reconstruction results

For infinitary theories in $\mathcal{L}_{\kappa,\kappa}$ for regular κ we have the following generalizations:

- (Adamek-Koubek-Velebil) κ-algebraic theories: Determined up to Cauchy completion by the functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits, κ-filtered colimits and regular epimorphisms.
- (Makkai-Pitts) κ-cartesian theories: Determined up to equivalence by functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits and κ-filtered colimits.
- (Makkai) κ-regular theories: Determined up to κ-exact completion by functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving products and κ-filtered colimits.

Reconstruction results

For infinitary theories in $\mathcal{L}_{\kappa,\kappa}$ for regular κ we have the following generalizations:

- (Adamek-Koubek-Velebil) κ-algebraic theories: Determined up to Cauchy completion by the functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits, κ-filtered colimits and regular epimorphisms.
- (Makkai-Pitts) κ-cartesian theories: Determined up to equivalence by functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits and κ-filtered colimits.
- (Makkai) κ-regular theories: Determined up to κ-exact completion by functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving products and κ-filtered colimits.

What we would like to do now is to find the generalization for the coherent case working with κ-coherent theories.

Reconstruction results

For infinitary theories in $\mathcal{L}_{\kappa,\kappa}$ for regular κ we have the following generalizations:

- (Adamek-Koubek-Velebil) κ-algebraic theories: Determined up to Cauchy completion by the functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits, κ-filtered colimits and regular epimorphisms.
- (Makkai-Pitts) κ-cartesian theories: Determined up to equivalence by functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving limits and κ-filtered colimits.
- (Makkai) κ-regular theories: Determined up to κ-exact completion by functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving products and κ-filtered colimits.

What we would like to do now is to find the generalization for the coherent case working with κ-coherent theories. These are the basic theories in $\mathcal{L}_{\kappa,\kappa}$ (Adámek-Rosický).

Reconstruction results

Two large cardinal assumptions are needed:

Reconstruction results

Two large cardinal assumptions are needed:

- κ needs to be strongly compact (every κ-complete filter can be extended to a κ-complete ultrafilter). This is needed since such a reconstruction result contains implicitely a completeness theorem.

Reconstruction results

Two large cardinal assumptions are needed:

- κ needs to be strongly compact (every κ-complete filter can be extended to a κ-complete ultrafilter). This is needed since such a reconstruction result contains implicitely a completeness theorem.
- The class of ordinals *Ord* is weakly compact: every tree whose height is a proper class and whose levels are set-sized has a cofinal branch (of a proper class size). This will allow to work with the proper class of models of the theory.

Reconstruction results

Two large cardinal assumptions are needed:

- κ needs to be strongly compact (every κ-complete filter can be extended to a κ-complete ultrafilter). This is needed since such a reconstruction result contains implicitely a completeness theorem.
- The class of ordinals *Ord* is weakly compact: every tree whose height is a proper class and whose levels are set-sized has a cofinal branch (of a proper class size). This will allow to work with the proper class of models of the theory.

One way to get the latter is to simply work with a Grothendieck universe V_λ for a weakly compact λ.

Reconstruction results

Two large cardinal assumptions are needed:

- κ needs to be strongly compact (every κ-complete filter can be extended to a κ-complete ultrafilter). This is needed since such a reconstruction result contains implicitely a completeness theorem.
- The class of ordinals *Ord* is weakly compact: every tree whose height is a proper class and whose levels are set-sized has a cofinal branch (of a proper class size). This will allow to work with the proper class of models of the theory.

One way to get the latter is to simply work with a Grothendieck universe V_λ for a weakly compact λ. But its exact strength over *ZFC* is equivalent to:

$$\{\exists \lambda (\lambda \text{ is n-Mahlo and } V_\lambda \preccurlyeq_{\Sigma_n} V) : n \in \omega\}$$

in the sense that the theory of classes $NBG + Ord$ is weakly compact proves ψ if and only if *ZFC* plus the above schema proves the relativization ψ^V.

Reconstruction results

The reconstruction result we propose looks as follows (the terminology will be explained later):

Reconstruction results

The reconstruction result we propose looks as follows (the terminology will be explained later):

Theorem

Assume that Ord is weakly compact and let κ be a strongly compact cardinal. Then κ-coherent theories can be recovered up to κ-pretopos completion as the accessible functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving κ-ultraproducts and κ-ultramorphisms with natural transformations that are compatible with κ-ultraproducts.

Reconstruction results

The reconstruction result we propose looks as follows (the terminology will be explained later):

Theorem

Assume that Ord is weakly compact and let κ be a strongly compact cardinal. Then κ-coherent theories can be recovered up to κ-pretopos completion as the accessible functors $Mod(\mathbf{T}) \to \mathcal{S}et$ preserving κ-ultraproducts and κ-ultramorphisms with natural transformations that are compatible with κ-ultraproducts.

Notation: categories with κ-ultraproducts and κ-ultramorphisms are called κ-ultracategories. The chief example is $\mathcal{S}et$, but also categories of models are κ-ultracategories.

κ-pretopos

Consider a κ-chain in a category $\mathcal{C}$ with κ-limits, i.e., a diagram $\Gamma : \gamma^{op} \to \mathcal{C}$ specified by morphisms $(h_{\beta,\alpha} : C_\beta \to C_\alpha)_{\alpha \leq \beta < \gamma}$ with the following condition:

κ-pretopos

Consider a κ-chain in a category $\mathcal{C}$ with κ-limits, i.e., a diagram $\Gamma : \gamma^{op} \to \mathcal{C}$ specified by morphisms $(h_{\beta,\alpha} : C_\beta \to C_\alpha)_{\alpha \leq \beta < \gamma}$ with the following condition:

- the restriction $\Gamma|_\beta$ is a limit diagram for every limit ordinal β.

κ-pretopos

Consider a κ-chain in a category $\mathcal{C}$ with κ-limits, i.e., a diagram $\Gamma : \gamma^{op} \to \mathcal{C}$ specified by morphisms $(h_{\beta,\alpha} : C_\beta \to C_\alpha)_{\alpha \leq \beta < \gamma}$ with the following condition:

- the restriction $\Gamma|_\beta$ is a limit diagram for every limit ordinal β.

We say that the morphisms $h_{\beta,\alpha}$ compose transfinitely, and take the limit projection $f_{\beta,0}$ to be the transfinite composite of $h_{\alpha+1,\alpha}$ for $\alpha < \beta$.

κ-pretopos

Consider a κ-chain in a category $\mathcal{C}$ with κ-limits, i.e., a diagram $\Gamma : \gamma^{op} \to \mathcal{C}$ specified by morphisms $(h_{\beta,\alpha} : C_\beta \to C_\alpha)_{\alpha \leq \beta < \gamma}$ with the following condition:

- the restriction $\Gamma|_\beta$ is a limit diagram for every limit ordinal β.

We say that the morphisms $h_{\beta,\alpha}$ compose transfinitely, and take the limit projection $f_{\beta,0}$ to be the transfinite composite of $h_{\alpha+1,\alpha}$ for $\alpha < \beta$. There is an exactness condition on $\mathcal{C}$ that we call *transfinite transitivity*: if we have a κ-tree of morphisms of $\mathcal{C}$ where the immediate successors of every node form a jointly covering family, then the transfinite composites of the morphisms along all possible cofinal branches of the tree forms itself a jointly covering family.

κ-pretopos

We have now:

κ-pretopos

We have now:

Definition

A κ-pretopos is a pretopos with κ-small limits, stable disjoint κ-small coproducts and that satisfies the transfinite transitivity property, i.e., the transfinite composites up to κ of jointly covering families of morphisms is itself jointly covering.

κ-pretopos

We have now:

Definition

A κ-pretopos is a pretopos with κ-small limits, stable disjoint κ-small coproducts and that satisfies the transfinite transitivity property, i.e., the transfinite composites up to κ of jointly covering families of morphisms is itself jointly covering.

The κ-pretopos satisfies a conceptual completeness theorem:

κ-pretopos

We have now:

Definition

A κ-pretopos is a pretopos with κ-small limits, stable disjoint κ-small coproducts and that satisfies the transfinite transitivity property, i.e., the transfinite composites up to κ of jointly covering families of morphisms is itself jointly covering.

The κ-pretopos satisfies a conceptual completeness theorem:

Theorem

If κ is strongly compact, there exists a universal κ-coherent solution to the problem of extending a category with κ-coproducts and quotients of equivalence relations, that we call the κ-pretopos completion. Moreover, if a κ-coherent functor $I : \mathcal{P} \to \mathcal{S}$ between κ-pretoposes induces an equivalence between their categories of models $I^* : Mod(\mathcal{S}) \to Mod(\mathcal{P})$, then I is itself an equivalence.

κ-ultraproducts

Given a set I and a κ-complete ultrafilter U on I (existing when κ is strongly compact), we have the following functor $[U]$ associated with U:

$$\mathcal{S}et^I \longrightarrow \mathcal{S}et$$

$$(A_i : i \in I) \mapsto \frac{\prod_{i \in I} A_i}{U}$$

κ-ultraproducts

Given a set I and a κ-complete ultrafilter U on I (existing when κ is strongly compact), we have the following functor $[U]$ associated with U:

$$\mathcal{S}et^I \longrightarrow \mathcal{S}et$$

$$(A_i : i \in I) \mapsto \frac{\prod_{i \in I} A_i}{U}$$

Then we have a categorical version of Łoś theorem:

κ-ultraproducts

Given a set I and a κ-complete ultrafilter U on I (existing when κ is strongly compact), we have the following functor $[U]$ associated with U:

$$\mathcal{S}et^I \longrightarrow \mathcal{S}et$$

$$(A_i : i \in I) \mapsto \frac{\prod_{i \in I} A_i}{U}$$

Then we have a categorical version of Łoś theorem:

Theorem

If κ is strongly compact and U is any κ-complete ultrafilter on I, the functor $[U]$ preserves the κ-pretopos structure.

κ-ultramorphisms

A κ-ultragraph Γ is given by:

κ-ultramorphisms

A κ-ultragraph Γ is given by:

κ-ultramorphisms

A κ-ultragraph Γ is given by:

- Two disjoint sets of nodes Γ^f (free nodes) and Γ^b (bound nodes)

κ-ultramorphisms

A κ-ultragraph Γ is given by:

- Two disjoint sets of nodes Γ^f (free nodes) and Γ^b (bound nodes)
- Edges $E(\gamma, \gamma')$ for pair of nodes γ, γ'

κ-ultramorphisms

A κ-ultragraph Γ is given by:

- Two disjoint sets of nodes Γ^f (free nodes) and Γ^b (bound nodes)
- Edges $E(\gamma, \gamma')$ for pair of nodes γ, γ'
- An assignment of a triple $(I_\beta, U_\beta, g_\beta)$ for each bound node β where U_β is a κ-complete ultrafilter on I_β and $g_\beta : I_\beta \to \Gamma^f$

κ-ultramorphisms

A κ-ultragraph Γ is given by:

- Two disjoint sets of nodes Γ^f (free nodes) and Γ^b (bound nodes)
- Edges $E(\gamma, \gamma')$ for pair of nodes γ, γ'
- An assignment of a triple $(I_\beta, U_\beta, g_\beta)$ for each bound node β where U_β is a κ-complete ultrafilter on I_β and $g_\beta : I_\beta \to \Gamma^f$

A κ-ultradiagram is a diagram $\mathcal{A} : \Gamma \to \mathcal{S}et$ satisfying $\mathcal{A}(\beta) = \frac{\prod_{i \in I_\beta} \mathcal{A}(g_\beta(i))}{U_\beta}$

κ-ultramorphisms

A κ-ultragraph Γ is given by:

- Two disjoint sets of nodes Γ^f (free nodes) and Γ^b (bound nodes)
- Edges $E(\gamma, \gamma')$ for pair of nodes γ, γ'
- An assignment of a triple $(I_\beta, U_\beta, g_\beta)$ for each bound node β where U_β is a κ-complete ultrafilter on I_β and $g_\beta : I_\beta \to \Gamma^f$

A κ-ultradiagram is a diagram $\mathcal{A} : \Gamma \to \mathcal{S}et$ satisfying $\mathcal{A}(\beta) = \frac{\Pi_{i \in I_\beta} \mathcal{A}(g_\beta(i))}{U_\beta}$
A κ-ultramorphism (Γ, k, l) is a natural transformation between the evaluation functors $ev_k, ev_l : Hom(\Gamma, \mathcal{S}et) \to \mathcal{S}et$ for given nodes k, l.

Reconstruction theorem

Facts:

Reconstruction theorem

Facts:

- Given a small κ-pretopos $\mathcal{T}$, the category of κ-pretopos morphisms to $\mathcal{S}et$ is an accessible κ-ultracategory, denoted $\mathbf{Hom}(\mathcal{T}, \mathcal{S}et)$.

Reconstruction theorem

Facts:

- Given a small κ-pretopos $\mathcal{T}$, the category of κ-pretopos morphisms to $\mathcal{S}et$ is an accessible κ-ultracategory, denoted $\mathbf{Hom}(\mathcal{T}, \mathcal{S}et)$.
- The category of accessible κ-ultrafunctors from $\mathbf{Hom}(\mathcal{T}, \mathcal{S}et)$ to $\mathcal{S}et$ with κ-ultratransformations is a κ-pretopos, denoted $\mathcal{H}om(\mathbf{Hom}(\mathcal{T}, \mathcal{S}et), \mathbf{Set})$.

Reconstruction theorem

Facts:

- Given a small κ-pretopos $\mathcal{T}$, the category of κ-pretopos morphisms to $\mathcal{S}et$ is an accessible κ-ultracategory, denoted $\mathbf{Hom}(\mathcal{T}, \mathcal{S}et)$.
- The category of accessible κ-ultrafunctors from $\mathbf{Hom}(\mathcal{T}, \mathcal{S}et)$ to $\mathcal{S}et$ with κ-ultratransformations is a κ-pretopos, denoted $\mathcal{H}om(\mathbf{Hom}(\mathcal{T}, \mathcal{S}et), \mathbf{Set})$.
- There is an evaluation functor $ev_{\mathcal{T}} : \mathcal{T} \to \mathcal{H}om(\mathbf{Hom}(\mathcal{T}, \mathcal{S}et), \mathbf{Set})$ which preserves the κ-pretopos structure.

Reconstruction theorem

Facts:

- Given a small κ-pretopos $\mathcal{T}$, the category of κ-pretopos morphisms to $\mathcal{S}et$ is an accessible κ-ultracategory, denoted $\mathbf{Hom}(\mathcal{T}, \mathcal{S}et)$.
- The category of accessible κ-ultrafunctors from $\mathbf{Hom}(\mathcal{T}, \mathcal{S}et)$ to $\mathcal{S}et$ with κ-ultratransformations is a κ-pretopos, denoted $\mathcal{H}om(\mathbf{Hom}(\mathcal{T}, \mathcal{S}et), \mathbf{Set})$.
- There is an evaluation functor $ev_{\mathcal{T}} : \mathcal{T} \to \mathcal{H}om(\mathbf{Hom}(\mathcal{T}, \mathcal{S}et), \mathbf{Set})$ which preserves the κ-pretopos structure.

Theorem

$ev_{\mathcal{T}}$ is an equivalence of categories.

Reconstruction theorem

The proof has three steps:

Reconstruction theorem

The proof has three steps:

- $ev_{\mathcal{T}}$ is conservative

Reconstruction theorem

The proof has three steps:

- $ev_{\mathcal{T}}$ is conservative
- $ev_{\mathcal{T}}$ is full on subobjects

Reconstruction theorem

The proof has three steps:

- $ev_{\mathcal{T}}$ is conservative
- $ev_{\mathcal{T}}$ is full on subobjects
- Every object of $\mathcal{H}om(\mathbf{Hom}(\mathcal{T}, \mathcal{S}et), \mathbf{Set})$ is κ-covered via $ev_{\mathcal{T}}$

Step 1: $ev_{\mathcal{T}}$ is conservative

That

$$ev_{\mathcal{T}} : \mathcal{T} \to \mathcal{H}om(\mathbf{Hom}(\mathcal{T}, \mathcal{S}et), \mathbf{Set})$$

is conservative is essentially a completeness theorem. It implies Karp's completeness theorem for infinitary classical logic, but is stronger, as it is a completeness theorem for κ-coherent theories.

Step 1: $ev_{\mathcal{T}}$ is conservative

That

$$ev_{\mathcal{T}} : \mathcal{T} \to \mathcal{H}om(\mathbf{Hom}(\mathcal{T}, \mathcal{S}et), \mathbf{Set})$$

is conservative is essentially a completeness theorem. It implies Karp's completeness theorem for infinitary classical logic, but is stronger, as it is a completeness theorem for κ-coherent theories.
This is related to an infinitary version of Deligne's completeness theorem: a κ-coherent topos (i.e., a topos where the coverage is generated by κ-small families and has the transfinite transitivity property) has enough κ-points (i.e., points whose inverse image preserve all κ-small limits).

Step 1: $ev_{\mathcal{T}}$ is conservative

That

$$ev_{\mathcal{T}} : \mathcal{T} \to \mathcal{H}om(\mathbf{Hom}(\mathcal{T}, \mathcal{S}et), \mathbf{Set})$$

is conservative is essentially a completeness theorem. It implies Karp's completeness theorem for infinitary classical logic, but is stronger, as it is a completeness theorem for κ-coherent theories.

This is related to an infinitary version of Deligne's completeness theorem: a κ-coherent topos (i.e., a topos where the coverage is generated by κ-small families and has the transfinite transitivity property) has enough κ-points (i.e., points whose inverse image preserve all κ-small limits).

Such a topos occurs as the κ-classifying topos of a κ-coherent theory, the universal property being that models of the theory in a topos with the transfinite transitivity property correspond to geometric morphisms from the topos whose inverse image preserve κ-small limits.

Step 2: $ev_{\mathcal{T}}$ is full on subobjects

This is the easiest step, and is the straightforward generalization of the same proof for the finitary case.

Step 2: $ev_{\mathcal{T}}$ is full on subobjects

This is the easiest step, and is the straightforward generalization of the same proof for the finitary case.
It is related to Beth's definability theorem, and in fact one can derive from it a version of definability for κ-coherent logic .

Step 3: κ-covering via $ev_{\mathcal{T}}$

One needs a reformulation of the notion of subobject in the category $\mathcal{H}om(\mathbf{Hom}(\mathcal{T}, \mathcal{S}et), \mathbf{Set})$ in terms of a transfinite construction involving κ-ultragraphs.

Step 3: κ-covering via $ev_{\mathcal{T}}$

One needs a reformulation of the notion of subobject in the category $\mathcal{H}om(\mathbf{Hom}(\mathcal{T}, \mathcal{S}et), \mathbf{Set})$ in terms of a transfinite construction involving κ-ultragraphs.

The crucial lemma needs Vopěnka's principle because we restricted ourselves to the accessible functors. In view of Step 2, it can be avoided for the final result.

Step 3: κ-covering via $ev_{\mathcal{T}}$

One needs a reformulation of the notion of subobject in the category $\mathcal{H}om(\mathbf{Hom}(\mathcal{T}, \mathcal{S}et), \mathbf{Set})$ in terms of a transfinite construction involving κ-ultragraphs.

The crucial lemma needs Vopěnka's principle because we restricted ourselves to the accessible functors. In view of Step 2, it can be avoided for the final result.

In the finitary case, one approximates the (reformulated) subobject taking a small subcategory of $\mathbf{Hom}(\mathcal{T}, \mathcal{S}et)$ and patches up the approximations using Keisler-Shelah isomorphism theorem (two elementarily equivalent models have isomorphic ultrapowers).

Step 3: κ-covering via $ev_{\mathcal{T}}$

Unfortunately Keisler-Shelah version for infinitary theories, using κ-complete ultrafilters, is false.

Step 3: κ-covering via $ev_{\mathcal{T}}$

Unfortunately Keisler-Shelah version for infinitary theories, using κ-complete ultrafilters, is false.
Counterexample (JDH): Pick $\alpha < \beta$ ordinals such that as linear orders they are elementarily equivalent. Then for any κ-complete ultrafilter U on I there is an elementary embedding:

$$j : V \to V^U$$

and we have:

$$\alpha^U \cong (j(\alpha), <) \neq (j(\beta), <) = \beta^U$$

Step 3: κ-covering via $ev_{\mathcal{T}}$

Unfortunately Keisler-Shelah version for infinitary theories, using κ-complete ultrafilters, is false.
Counterexample (JDH): Pick $\alpha < \beta$ ordinals such that as linear orders they are elementarily equivalent. Then for any κ-complete ultrafilter U on I there is an elementary embedding:

$$j : V \to V^U$$

and we have:

$$\alpha^U \cong (j(\alpha), <) \neq (j(\beta), <) = \beta^U$$

In place of Keisler-Shelah, we use instead that *Ord* is weakly compact.

Stone duality

Let $\mathbf{AccUlt}_\kappa$ be the 2-category of accessible κ-ultracategories with accessible κ-ultrafunctors and κ-ultratransformations.

Stone duality

Let $\mathbf{AccUlt}_\kappa$ be the 2-category of accessible κ-ultracategories with accessible κ-ultrafunctors and κ-ultratransformations.

Theorem

Let κ be strongly compact, and assume that Ord is weakly compact. There is an equivalence (given by homming into $\mathcal{S}et$) between the category of small κ-pretoposes and a full subcategory of $\mathbf{AccUlt}_\kappa$ whose objects are themselves equivalent to categories of models of first-order theories in $\mathcal{L}_{\kappa,\kappa}$.

Stone duality

Let $\mathbf{AccUlt}_\kappa$ be the 2-category of accessible κ-ultracategories with accessible κ-ultrafunctors and κ-ultratransformations.

Theorem

Let κ be strongly compact, and assume that Ord is weakly compact. There is an equivalence (given by homming into $\mathcal{S}et$) between the category of small κ-pretoposes and a full subcategory of $\mathbf{AccUlt}_\kappa$ whose objects are themselves equivalent to categories of models of first-order theories in $\mathcal{L}_{\kappa,\kappa}$.

Corollary

The category of κ-pretopos morphisms between two small κ-pretoposes $\mathcal{T}$ and $\mathcal{S}$ is equivalent to the category of accessible κ-ultrafunctors between $Mod(\mathcal{T})$ and $Mod(\mathcal{S})$.

Future work

As further lines of work to pursue, we have:

Future work

As further lines of work to pursue, we have:

- Generalize to the infinitary case the duality theorem of Awodey-Forssell by working with κ-pretoposes. This will provide a reconstruction result of a topological flavour that does not need *Ord* to be weakly compact. It does require, however, the compactness of κ.

Future work

As further lines of work to pursue, we have:

- Generalize to the infinitary case the duality theorem of Awodey-Forssell by working with κ-pretoposes. This will provide a reconstruction result of a topological flavour that does not need *Ord* to be weakly compact. It does require, however, the compactness of κ.
- Extend Makkai's duality theory for Boolean pretoposes setup to the infinitary case, when possible.

Future work

As further lines of work to pursue, we have:

- Generalize to the infinitary case the duality theorem of Awodey-Forssell by working with κ-pretoposes. This will provide a reconstruction result of a topological flavour that does not need *Ord* to be weakly compact. It does require, however, the compactness of κ.
- Extend Makkai's duality theory for Boolean pretoposes setup to the infinitary case, when possible.
- Use Zawadowski's argument for proving descent, based on this duality.

Future work

As further lines of work to pursue, we have:

- Generalize to the infinitary case the duality theorem of Awodey-Forssell by working with κ-pretoposes. This will provide a reconstruction result of a topological flavour that does not need *Ord* to be weakly compact. It does require, however, the compactness of κ.
- Extend Makkai's duality theory for Boolean pretoposes setup to the infinitary case, when possible.
- Use Zawadowski's argument for proving descent, based on this duality.

Thank you!