

Weak saturation and weak amalgamation property

Ivan Di Liberti
PSSL, Leeds 2017

Theorem (Fraïssé '54)

Let \mathcal{K} be a class of finite structures in a signature Σ such that:

Theorem (Fraïssé '54)

Let \mathcal{K} be a class of finite structures in a signature Σ such that:

- is closed under isomorphism;

Theorem (Fraïssé '54)

Let \mathcal{K} be a class of finite structures in a signature Σ such that:

- is closed under isomorphism;
- is closed under substructures;

Theorem (Fraïssé '54)

Let \mathcal{K} be a class of finite structures in a signature Σ such that:

- is closed under isomorphism;
- is closed under substructures;
- has only countably many members up to isomorphism;

Theorem (Fraïssé '54)

Let \mathcal{K} be a class of finite structures in a signature Σ such that:

- is closed under isomorphism;
- is closed under substructures;
- has only countably many members up to isomorphism;
- has the amalgamation property;

Theorem (Fraïssé '54)

Let \mathcal{K} be a class of finite structures in a signature Σ such that:

- is closed under isomorphism;
- is closed under substructures;
- has only countably many members up to isomorphism;
- has the amalgamation property;
- has the joint embedding property;

Theorem (Fraïssé '54)

Let \mathcal{K} be a class of finite structures in a signature Σ such that:

- is closed under isomorphism;
- is closed under substructures;
- has only countably many members up to isomorphism;
- has the amalgamation property;
- has the joint embedding property;

then there is a structure M which is:

Theorem (Fraïssé '54)

Let \mathcal{K} be a class of finite structures in a signature Σ such that:

- is closed under isomorphism;
- is closed under substructures;
- has only countably many members up to isomorphism;
- has the amalgamation property;
- has the joint embedding property;

then there is a structure M which is:

- finitely saturated;

Theorem (Fraïssé '54)

Let \mathcal{K} be a class of finite structures in a signature Σ such that:

- is closed under isomorphism;
- is closed under substructures;
- has only countably many members up to isomorphism;
- has the amalgamation property;
- has the joint embedding property;

then there is a structure M which is:

- finitely saturated;
- finitely homogeneous;

Theorem (Fraïssé '54)

Let \mathcal{K} be a class of finite structures in a signature Σ such that:

- is closed under isomorphism;
- is closed under substructures;
- has only countably many members up to isomorphism;
- has the amalgamation property;
- has the joint embedding property;

then there is a structure M which is:

- finitely saturated;
- finitely homogeneous;
- countable;

Theorem (Fraïssé '54)

Let \mathcal{K} be a class of finite structures in a signature Σ such that:

- is closed under isomorphism;
- is closed under substructures;
- has only countably many members up to isomorphism;
- has the amalgamation property;
- has the joint embedding property;

then there is a structure M which is:

- finitely saturated;
- finitely homogeneous;
- countable;
- unique up to isomorphism between structures having these properties above.

Theorem (Rosický '97)

Theorem (Rosický '97)

Let \mathcal{K} be a λ -accessible category such that:

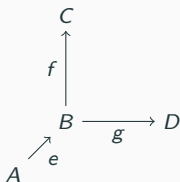
- has directed colimits;
- has the amalgamation property;
- has the joint embedding property;

then there is an object K which is:

- λ -saturated;
- λ -homogeneous;
- if it is λ^+ presentable, then it is unique up to isomorphism.

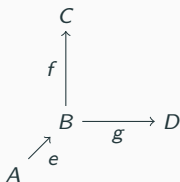
Definition

A category \mathcal{K} has the weak amalgamation property if, for any object A there is an arrow $A \xrightarrow{e} B$ such that any span like the following



Definition

A category \mathcal{K} has the weak amalgamation property if, for any object A there is an arrow $A \xrightarrow{e} B$ such that any span like the following



can be completed to a square such that the diagram below is commutative.



We call such an arrow $A \xrightarrow{e} B$ *amalgamable*.

Motivation (Ivanov and Kechris-Rosendal)

Motivation (Ivanov and Kechris-Rosendal)

To each Fraïssé class \mathcal{K} one can associate the class \mathcal{K}_p of all systems, $S = (A, \psi : B \rightarrow C)$ where A, B, C are structures in \mathcal{K} , B and C are substructures of A . Moreover, ψ is an isomorphism.

Theorem (Kechris-Rosendal)

The following are equivalent:

- \mathcal{K}_p has the weak amalgamation property and the joint embedding property;
- the Fraïssé limit of \mathcal{K} has a generic automorphism.

Motivation (Ivanov and Kechris-Rosendal)

To each Fraïssé class \mathcal{K} one can associate the class \mathcal{K}_p of all systems, $S = (A, \psi : B \rightarrow C)$ where A, B, C are structures in \mathcal{K} , B and C are substructures of A . Moreover, ψ is an isomorphism.

Theorem (Kechris-Rosendal)

The following are equivalent:

- \mathcal{K}_p has the weak amalgamation property and the joint embedding property;
- the Fraïssé limit of \mathcal{K} has a generic automorphism.

Examples

Many class of graphs.

Motivation (Ivanov and Kechris-Rosendal)

To each Fraïssé class \mathcal{K} one can associate the class \mathcal{K}_p of all systems, $S = (A, \psi : B \rightarrow C)$ where A, B, C are structures in \mathcal{K} , B and C are substructures of A . Moreover, ψ is an isomorphism.

Theorem (Kechris-Rosendal)

The following are equivalent:

- \mathcal{K}_p has the weak amalgamation property and the joint embedding property;
- the Fraïssé limit of \mathcal{K} has a generic automorphism.

Examples

Many class of graphs.

A concrete example

The class of all finite cycle-free graphs in which no two vertices of degree > 2 are adjacent.

Definition

An object K is weakly λ -saturated when for any arrow $A \rightarrow K$ where A is λ -presentable there exists $A \rightarrow B$, with B λ -presentable such that for any prolongation $A \rightarrow B \rightarrow C$ where C is λ -presentable there is an arrow $C \rightarrow K$ making the obvious diagram commutative.

Kubiś '17

Kubiś '17

Let $\mathcal{K} \subset \mathcal{L}$ be two categories such that:

Kubiś '17

Let $\mathcal{K} \subset \mathcal{L}$ be two categories such that:

- \mathcal{K} has the joint embedding property, the weak amalgamation property, is weakly dominated by a countable subcategory;

Kubiś '17

Let $\mathcal{K} \subset \mathcal{L}$ be two categories such that:

- \mathcal{K} has the joint embedding property, the weak amalgamation property, is weakly dominated by a countable subcategory;
- every arrow in \mathcal{L} is monic;

Kubiś '17

Let $\mathcal{K} \subset \mathcal{L}$ be two categories such that:

- \mathcal{K} has the joint embedding property, the weak amalgamation property, is weakly dominated by a countable subcategory;
- every arrow in \mathcal{L} is monic;
- every directed diagram in \mathcal{K} has a colimit in \mathcal{L} and every object in \mathcal{L} is the directed colimit of a diagram in \mathcal{K} ;

Kubiś '17

Let $\mathcal{K} \subset \mathcal{L}$ be two categories such that:

- \mathcal{K} has the joint embedding property, the weak amalgamation property, is weakly dominated by a countable subcategory;
- every arrow in \mathcal{L} is monic;
- every directed diagram in \mathcal{K} has a colimit in \mathcal{L} and every object in \mathcal{L} is the directed colimit of a diagram in \mathcal{K} ;
- every object in \mathcal{K} is ω -presentable in \mathcal{L} .

Kubiś '17

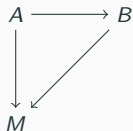
Let $\mathcal{K} \subset \mathcal{L}$ be two categories such that:

- \mathcal{K} has the joint embedding property, the weak amalgamation property, is weakly dominated by a countable subcategory;
- every arrow in \mathcal{L} is monic;
- every directed diagram in \mathcal{K} has a colimit in \mathcal{L} and every object in \mathcal{L} is the directed colimit of a diagram in \mathcal{K} ;
- every object in \mathcal{K} is ω -presentable in \mathcal{L} .

then there is a weakly finitely saturated object which is also weakly finitely homogeneous and unique up to isomorphism.

Definition

We say that a category \mathcal{K} with the weak amalgamation property satisfies the smallness condition if, given a λ -presentable object A and an amalgamable arrow $A \rightarrow M$, there exists a λ -presentable object B and arrows $A \rightarrow B, B \rightarrow M$ such that $A \rightarrow B$ is amalgamable and the diagram below commutes.



Theorem

Let \mathcal{K} be a λ -accessible category with the weak amalgamation property and directed colimits, satisfying the smallness condition, then any object K has a map $K \rightarrow M$ where M is weakly λ -saturated.

Theorem

Let \mathcal{K} be a λ -accessible category with the weak amalgamation property and directed colimits, satisfying the smallness condition, then any object K has a map $K \rightarrow M$ where M is weakly λ -saturated.

In this result Rosický uses the characterization of saturated objects as closed ones. Unfortunately we did not find a notion of weakly closed object.

Theorem

Let \mathcal{K} be a λ -accessible category with the weak amalgamation property and directed colimits, satisfying the smallness condition, then any object K has a map $K \rightarrow M$ where M is weakly λ -saturated.

In this result Rosický uses the characterization of saturated objects as closed ones. Unfortunately we did not find a notion of weakly closed object.

Theorem

Let \mathcal{K} be a λ -accessible category having directed colimits and the joint embedding property. Then any two weakly λ -saturated, λ^+ -presentable objects are isomorphic.

Theorem

Let \mathcal{K} be a λ -accessible category with the weak amalgamation property and directed colimits, satisfying the smallness condition, then any object K has a map $K \rightarrow M$ where M is weakly λ -saturated.

In this result Rosický uses the characterization of saturated objects as closed ones. Unfortunately we did not find a notion of weakly closed object.

Theorem

Let \mathcal{K} be a λ -accessible category having directed colimits and the joint embedding property. Then any two weakly λ -saturated, λ^+ -presentable objects are isomorphic.

In this result Kubiś uses generic sequences, a tool that he has since there is a countable dominating subcategory.

Theorem

Let \mathcal{K} be a λ -accessible category with the weak amalgamation property and directed colimits, satisfying the smallness condition, then any object K has a map $K \rightarrow M$ where M is weakly λ -saturated.

In this result Rosický uses the characterization of saturated objects as closed ones. Unfortunately we did not find a notion of weakly closed object.

Theorem

Let \mathcal{K} be a λ -accessible category having directed colimits and the joint embedding property. Then any two weakly λ -saturated, λ^+ -presentable objects are isomorphic.

In this result Kubiś uses generic sequences, a tool that he has since there is a countable dominating subcategory.

Theorem

Let \mathcal{K} be a λ -accessible category having directed colimits and the joint embedding property. A weakly λ -saturated, λ^+ -presentable is weakly λ -homogeneous.