Homotopy Type Theory and Algebraic Model Structures (II)

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Current status

Last talk:

(1) algebraic weak factorization system $(\mathsf{Cof}, \mathsf{TrivFib})$,

(2) algebraic weak factorization system $(\mathsf{TrivCof}, \mathsf{Fib})$,

(3) $(\mathsf{TrivCof}, \mathsf{Fib})$ has the Frobenius property.

This talk:

(4) prove the glueing property,

(5) prove the fibration extension property,

(6) show that we have an algebraic model structure.

Setting (I)

Presheaf category $\mathcal{E}$.

Functorial cylinder $X \mapsto I \otimes X$, endpoint inclusions $\delta^k \otimes X \colon X \to I \otimes X$.

(C1) $I \otimes (-)$ has contractions,

(C2) $I \otimes (-)$ has connections,

(C3) $I \otimes (-)$ has a right adjoint $\hom(I, -)$,

(C4) $I \otimes (-) \colon \mathcal{E} \to \mathcal{E}$ preserves pullback squares,

(C5) the endpoint inclusions $\delta^k \otimes X \colon X \to I \otimes X$ are cartesian;

Remark.
All structure on the functorial cylinder $I \otimes (-)$ transposes to dual structure on the functorial cocylinder $\hom(I, -)$, e.g.:
endpoint projections $\hom(\delta^k, X) \colon \hom(I, X) \to X$.

Setting (II)

Full subcategory $\mathcal{M} \hookrightarrow \mathcal{E}^{\to}_{\mathsf{cart}}$ spanned by monomorphisms such that:

(M1) the unique map $\bot_X \colon 0 \to X$ is in $\mathcal{M}$, for every $X \in \mathcal{E}$,

(M2) $\mathcal{M}$ is closed under pullbacks,

(M3) $\mathcal{M}$ is closed under pushout product with the endpoint inclusions,

$\mathcal{M}$ generates the awfs $(\mathsf{Cof}, \mathsf{TrivFib})$.

$\{\delta^0, \delta^1\} \mathbin{\hat{\otimes}} \mathcal{M}$ generates the awfs $(\mathsf{TrivCof}, \mathsf{Fib})$.

Outline

To simplify the presentation, we first do things non-algebraically in the traditional style:

(4) prove the glueing property,

(5) prove the fibration extension property,

(6) show that we have a model structure.

In the last part, we will then treat some key points that appear in the algebraic setting, allowing us to conclude:

(7) we have an algebraic model structure.

Along the way, we will occasionally have to add a new assumption to our setting (C1–C5) and (M1–M3). This will be clearly indicated.

Notation

Before we can get going, we need to introduce some further notation.

Notation: path objects

Given a fibration $p\colon A \to B$, we have the **path object** factorization

$$A \xrightarrow{\ r\ } P_B A \xrightarrow{\langle d_0, d_1 \rangle} A \times_B A$$

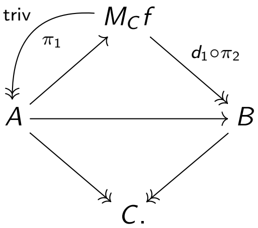
with d_0 and d_1 trivial fibrations where $P_B A$ is defined as follows:

$$\begin{array}{ccc}
P_B A & \longrightarrow & \hom(I, A) \\
\big\downarrow & \lrcorner & \big\downarrow {\scriptstyle \widehat{\hom}([\delta^0, \delta^1], p)} \\
A \times_B A & \longrightarrow & A \times_B \hom(I, B) \times_B A \\
\big\downarrow & \lrcorner & \big\downarrow \\
B & \xrightarrow{\hom(\epsilon, B)} & \hom(I, B).
\end{array}$$

($\widehat{\hom}$ denotes the pullback hom, adjoint to the pushout product $\hat{\otimes}$.)

Notation: mapping path spaces

Given a map $f\colon A \to B$ between objects fibrant over C,
we have the **mapping path space** factorization

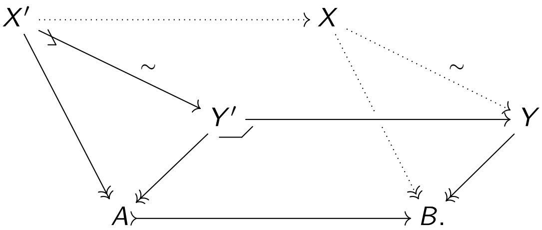


where $M_C f = A \times_B P_C B$.

f is a **homotopy equivalence** (over C) if $M_C f \to B$ is a trivial fibration.

Goal (4): glueing property

Recall the **glueing property**, stating that weak equivalences between fibrations extend along cofibrations:

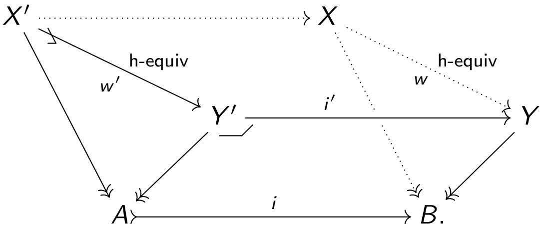


We don't yet have a notion of weak equivalence (will come with goal (6)).

Instead, we prove the glueing property for homotopy equivalences.
Every object is cofibrant (M1), so both statements will end up equivalent.

Goal (4): glueing property

Theorem (Glueing)**.**
Homotopy equivalences between fibrations extend along cofibrations:



Every object is cofibrant (M1), so both statements will end up equivalent.

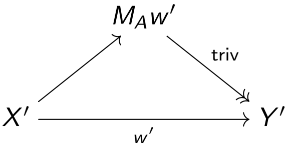
The proof we present is derived from Coquand et al.

Proof. We let $X =_{\mathsf{def}} i'_* X'$ be the pushforward of X' along i'.
Since i' is mono, all horizontal squares form pullbacks as desired.

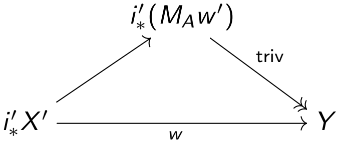
Critical: why is $X \to B$ a fibration?

Goal (4): glueing property (proof)

Since $w' \colon X' \to Y'$ is a homotopy equivalence over A, the second leg of the mapping path space factorization of w' will be a trivial fibration:



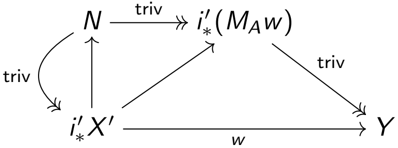
Since cofibrations are closed under pullback (M2), trivial fibrations are preserved under pushforward:



Goal (4): glueing property (proof)

Let us now work in the slice over B.

We claim (next slide) that $i'_* X' \to i'_*(M_A w)$ further factorizes into a section to a trivial fibration followed by a trivial fibration:



Since Y is fibrant, so is N and its retract $X \cong i'_* X'$.

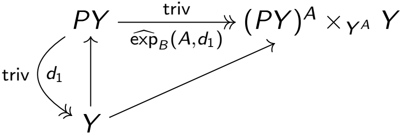
This also exhibits w as a homotopy equivalence.

Goal (4): glueing property (proof)

By definition, $X' \to M_A w$ is a pullback of $Y' \to P_A Y'$.

So instead we may verify that $i'_* Y' \to i'_* P_A Y'$ factorizes into a section to a trivial fibration followed by a trivial fibration.

But $i'_* Y' = Y$ and $i'_* P_A Y' = i_* P_A Y' \times_{i_* Y'} Y = (P_B Y)^A \times_{Y^A} Y$, so such a factorization is



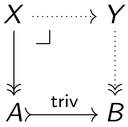
The top map is the pullback exponential (over B) of d_1 with $i \colon A \to B$.

To make it a trivial fibration, by adjointness we have to add the following assumption to our setting:

(M4) $\mathcal{M}$ is closed under binary union. □

Goal (5): fibration extension property

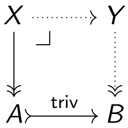
Recall the **fibration extension property**, stating that fibrations extend along trivial cofibrations:



Assuming fibrations are local (see previous talk),
it suffices to show this for generating trivial cofibrations.

Recall: cylinder inclusions are a generating class for trivial cofibrations.

Lemma.
We can extend fibrations $X \to A$ along cylinder inclusions $A \to B$:



Goal (5): fibration extension property (proof)

Proof.
By a lemma (previous talk), the cylinder inclusion $A \to B$ is a strong (say right) homotopy equivalence.

By a lemma (previous talk), the strong right homotopy equivalence $A \to B$ gives rise to a retract of arrows:

$$\begin{array}{ccccc}
\{0\} \otimes A & \longrightarrow & (I \otimes A) \cup (\{1\} \otimes B) & \dashrightarrow & \{0\} \otimes A \\
\downarrow & & \downarrow & & \downarrow \\
\{0\} \otimes B & \longrightarrow & I \otimes B & \dashrightarrow & \{0\} \otimes B
\end{array}$$

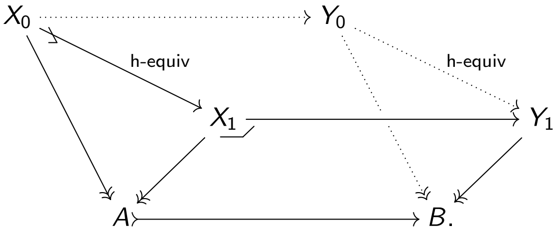
We start out with a fibration over $A \cong \{0\} \otimes A$.

We pull it back to a fibration over $(I \otimes A) \cup (\{1\} \otimes B)$.

Goal (5): fibration extension property (proof)

A fibration over $I \otimes A$ induces fibrant objects X_0 and X_1 over A and a homotopy equivalence between them (uses cofibrancy of objects (M1)).

Our fibration over $(I \otimes A) \cup (\{1\} \otimes B)$ thus induces input data for glueing:



Note that $X_0 \cong X$.

The resulting fibration $Y_0 \to B$ solves the fibration extension problem.

□

Goal (5): fibration extension property (proof)

From this point onwards, we assume that fibrations are local
(in the sense of Cisinski, see previous talk).

We may then conclude from our lemma the following.

Theorem. The fibration extension property holds. □

Remark.

(i) We lack a mechanism for putting homotopy equivalences between fibrant objects over B back together to a fibration over $I \otimes B$.

Hence we cannot apply the glueing property directly to get fibration extension along cylinder inclusions.

(ii) Our detour via strong homotopy equivalences is an example of Coquand et al.'s technique reducing Kan filling to Kan composition.

Goal (6): model structure

We now have the ingredients in hand to show that the weak factorization systems $(\mathsf{Cof}, \mathsf{TrivFib})$ and $(\mathsf{TrivCof}, \mathsf{Fib})$ form a model structure.

The definition of weak equivalence is forced upon us:

Definition. The class $\mathsf{Weq} \subseteq \mathcal{E}^{\to}$ of *weak equivalences* consists of all maps that factor as a trivial cofibration followed by a trivial fibration.

Lemma. We have:

(i) $\mathsf{Cof} \cap \mathsf{Weq} = \mathsf{TrivCof}$,

(ii) $\mathsf{Fib} \cap \mathsf{Weq} = \mathsf{TrivFib}$.

Proof. We have $\mathsf{TrivCof}, \mathsf{TrivFib} \subseteq \mathsf{Weq}$ for trivial reasons.

Standard retract arguments show that:

(i) $\mathsf{Cof} \cap \mathsf{Weq} \subseteq \mathsf{TrivCof}$,

(ii) $\mathsf{Fib} \cap \mathsf{Weq} \subseteq \mathsf{TrivFib}$. □

It remains to show that weak equivalences satisfy 2-out-of-3.

Goal (6): 2-out-of-3 for trivial fibrations among fibrations

Lemma. If two of the fibrations are trivial fibrations, then so is the third:

$$\begin{array}{ccc} & Y & \\ {\scriptstyle p}\nearrow & & \searrow{\scriptstyle q} \\ X & \xrightarrow[r]{} & Z \end{array}$$

Proof.
Assume p and r trivial. Then p has a section (as everything is cofibrant), exhibiting q as a retract of r. Since r is trivial, so is its retract q.

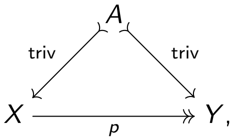
Assume q and r trivial. Then p is retract of the composite trivial fibration

$$\begin{array}{ccc} \hom(I, X) & \xrightarrow[\widehat{\hom}(\delta^0, p)]{\text{triv}} & X \times_Y \hom(I, Y) \\ & & \Big\downarrow {\scriptstyle \text{triv}\ \ X \times_Y \widehat{\hom}([\delta^0, \delta^1], q)} \\ \hom(I, Z) \times_Z Y & \xleftarrow[r \times_Z \hom(I, Z) \times_Z Y]{\text{triv}} & X \times_Z \hom(I, Z) \times_Z Y. \end{array}$$

□

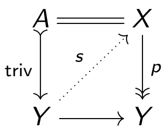
Goal (6): span property

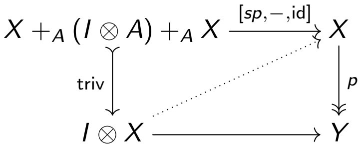
Lemma (Span property)**.** Given trivial cofibrations $A \to X$ and $A \to Y$ and a fibration $X \to Y$ commuting as below:



we have $p\colon X \to Y$ a trivial fibration.

Proof. We make p into a costrong deformation retract:

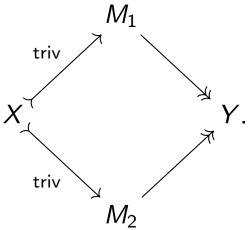




Trivial fibrations can be seen to coincide with fibrations that are costrong deformation retracts. □

Goal (6): factorization lemma

Lemma (Factorization lemma)**.**
Consider two (TrivCof, Fib)-factorizations of a map $X \to Y$:

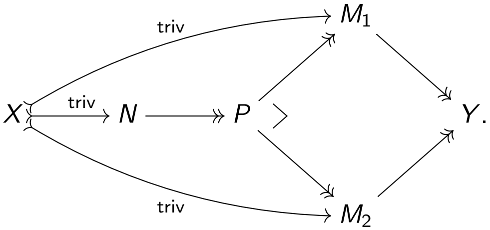


If $M_1 \to Y$ is a trivial fibration, then so is $M_2 \to Y$.

In particular, a map f is a weak equivalence if any *specific* (TrivCof, Fib)-factorization of f has trivial fibration part.

Goal (6): factorization lemma (proof)

Proof.
We draw the pullback P of $M_1 \to Y$ and $M_2 \to Y$ and factor the induced map $X \to P$ as a trivial cofibration followed by a fibration:

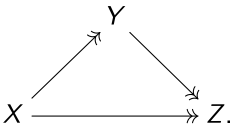


By the span property, $N \to M_1$ and $N \to M_2$ are trivial fibrations.

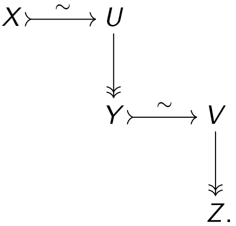
We then use stability of trivial fibrations under pullback and repeated 2-out-of-3 for trivial fibrations relative to fibrations. □

Goal (6) 2-out-of-3 for weak equivalences

Lemma. If two of the maps are weak equivalences, then so is the third:

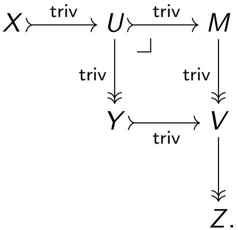


Proof. We start by factoring $X \to Y$ and $Y \to Z$ into trivial cofibrations followed by fibrations:



Goal (6): 2-out-of-3 for weak equivalences (easy cases)

Assume first that $X \to Y$ is a weak equivalence.
Then $U \to Y$ is a trivial fibration (factorization lemma).
We use pushforward to extend it along the mono $Y \to V$:



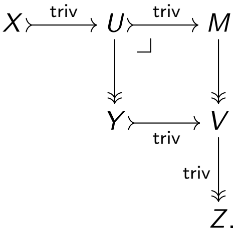
Note that $U \to M$ is a trivial cofibration by the **Frobenius property**.

By factorization lemma, 2-out-of-3 for trivial fibrations among fibrations:

$$X \to Z \text{ weak equivalence} \iff Y \to Z \text{ weak equivalence.}$$

Goal (6): 2-out-of-3 for weak equivalences (hard case)

Assume now that $X \to Z$ and $Y \to Z$ are weak equivalences.
We use the **fibration extension property** to extend the fibration $U \to Y$ along the trivial cofibration $Y \to V$.



By the factorization lemma, $M \to Z$ and $V \to Z$ are trivial fibrations.
By 2-out-of-3 for trivial fibrations among fibrations, so is $M \to V$.
Then $U \to Y$ is a trivial fibration, making $X \to Y$ a weak equivalence. □

Goal (6): model structure

Thus we have the following.

Theorem.
$(\mathrm{Weq}, \mathrm{Fib}, \mathrm{Cof})$ forms a proper model structure on $\mathcal{E}$.

Proof.
We have just verified that the weak equivalences Weq fit into the two weak factorization systems and satisfy 2-out-of-3.

Left properness follows from cofibrancy of all objects (M1).

Right properness was shown in the previous talk. □

Corollary. Simplicial sets form a proper model category.

Goal (7): algebraization

We have algebraic analogues of all previous properties, which posit the existence of certain functors involving the category of uniform fibrations.

However, there are two subtle points that require stronger assumptions:

(C6) representables are closed under $I \otimes (-)$.

(This replaces non-algebraic locality of fibrations in order to get an algebraic universe of fibrations.)

(M4) $\mathcal{M}$ is closed under $\hom(I, -)$.

Altogether, we get an *algebraic* model structure in a strong sense (due to Emily Riehl and Andrew Swan, next slide).

Goal (7): algebraic model structure

Definition (Algebraic model structure; Riehl, Swan)**.**
An *algebraic model structure* is a morphism

$$(\mathsf{TrivCof}, \mathsf{Fib}) \to (\mathsf{Cof}, \mathsf{TrivFib})$$

of algebraic weak factorization systems together with a category

$$\mathsf{Weq} \to \mathcal{E}^{\to}$$

of weak equivalences satisfying *functorial 2-out-of-3* and admitting maps over $\mathcal{E}^{\to}$ between:

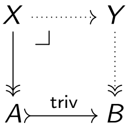
$\mathsf{TrivCof}$ and $\mathsf{Cof} \times_{\mathcal{E}^{\to}} \mathsf{Weq}$,

$\mathsf{TrivFib}$ and $\mathsf{Fib} \times_{\mathcal{E}^{\to}} \mathsf{Weq}$.

Every ams has an *underlying model structure*.

Goal (7): where is (M4) needed?

In an algebraic setting, functorially extending uniform fibrations



should also entail that the lifting structure for $Y \to B$ restricts to the lifting structure for $X \to A$.

It is not at all clear from our previous constructions why this should be possible.

Fortunately, we can fix it after the fact.

Goal (7): aligning lifting structures (I)

Lemma (Aligning uniform trivial fibrations)**.** In any square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle \text{triv}}\downarrow & \lrcorner & \downarrow{\scriptstyle \text{triv}} \\ A & \rightarrowtail & B, \end{array}$$

of uniform trivial fibrations, we can uniformly[1] replace the uniform trivial fibration structure on $Y \to B$ by one that makes the structures cohere.

Proof.
Use closure of $\mathcal{M}$ under binary union (M4).

[1]as in: a functor $(\mathsf{TrivFib} \times_{\mathcal{E}^{\to}} \mathcal{E}^{\to}_{\mathsf{cart}} \times_{\mathcal{E}^{\to}} \mathsf{TrivFib}) \times_{\mathcal{E}^{\to}} \mathsf{Cof} \to \mathsf{TrivFib}^{\to}$

Goal (7): aligning lifting structures (II)

Lemma (Aligning uniform fibrations)**.** In any square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow & \lrcorner & \big\downarrow \\ A & \rightarrowtail & B, \end{array}$$

of uniform fibrations, we can uniformly[2] replace the uniform fibration structure on $Y \to B$ by one that makes the structures cohere.

Proof.
Applying the functor $\widehat{\hom}(\delta^k, -)$ to both $X \to A$ and $Y \to B$ reduces the problem to aligning uniform trivial fibrations over a base change of $\hom(I, A) \to \hom(I, B)$, which is a cofibration by (M4).

[2]as in: a functor $(\mathsf{Fib} \times_{\mathcal{E}^\to} \mathcal{E}^\to_{\mathsf{cart}} \times_{\mathcal{E}^\to} \mathsf{Fib}) \times_{\mathcal{E}^\to} \mathsf{Cof} \to \mathsf{Fib}^\to$

Goal (7): the algebraic model structure

Thus we have the following.

Theorem.
(Weq, Fib, Cof) forms a proper algebraic model structure on $\mathcal{E}$. □

Corollary. Cubical sets as considered by Coquand et al. form a proper algebraic model category.

Note. This cube category (with diagonals, symmetries, and connections) is not a Reedy category. Also, not all monomorphims are cofibrations.

Question. Is this model category Quillen equivalent to the standard one on simplicial sets? Please tell me!