Homotopy Type Theory and Algebraic Model Structures (I)

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Plan of the talks

Goal

- analysis of the cubical set model of Homotopy Type Theory

By-products

- general method to obtain right proper algebraic model structures,
- a new proof of model structure for Kan complexes and its right properness, avoiding minimal fibrations.

Outline of this talk

Part I: Homotopy Type Theory

- Models of HoTT
- Quillen model structures
- Some issues

Part II: Uniform fibrations and the Frobenius property

- Algebraic weak factorization systems
- Uniform fibrations
- The Frobenius property

References

1. C. Cohen, T. Coquand, S. Huber and A. Mörtberg
   Cubical Type Theory: a constructive interpretation of the univalence axiom
   arXiv, 2016.

2. N. Gambino and C. Sattler
   Frobenius condition, right properness, and uniform fibrations
   arXiv, 2016.

Part I: Homotopy Type Theory

Homotopy Type Theory

HoTT = Martin-Löf's type theory + Voevodsky's univalence axiom

Key ingredients:

(1) substitution operation,
(2) identity types,
(3) Π-types,
(4) a type universe,
(5) univalence axiom.

We give a category-theoretic account of these.

Models of HoTT

Definition. A *model of homotopy type theory* consists of

- a category $\mathcal{E}$ with a terminal object 1,
- a class Fib of maps, called *fibrations*,

subject to axioms (1)-(5).

Idea: the sequent

$$x : A \vdash B(x) : \mathrm{type}$$

is interpreted as a fibration

$$\begin{tikzcd} B \arrow[d, "p"] \\ A \end{tikzcd}$$

Warning: issues of coherence will be ignored.

Models of HoTT: substitution

(1) Pullbacks of fibrations exist and are again fibrations.

So for every map $\sigma : A' \to A$ we have

$$\mathsf{Fib}/A \xrightarrow{\sigma^*} \mathsf{Fib}/A'$$

Diagrammatically:

$$\begin{tikzcd} B' \arrow[r] \arrow[d, "p'"'] & B \arrow[d, "p"] \\ A' \arrow[r, "\sigma"'] & A \end{tikzcd}$$

Models of HoTT: identity types

(2) For every fibration $p : B \to A$, there is a factorization

$$\begin{tikzcd} B \arrow[r, "r"] \arrow[dr, bend right, "\Delta_p"'] & \mathrm{Id}_B \arrow[d, "q"] \\ & B \times_A B \end{tikzcd}$$

where $r \in {}^{\pitchfork}\mathrm{Fib}$ and $q \in \mathrm{Fib}$.

Models of HoTT: Π-types

(3) If $p : B \to A$ is a fibration, pullback along p has a right adjoint

$$\mathrm{Fib}/B \xrightarrow{p_*} \mathrm{Fib}/A$$

We call this the *pushforward* along p.

Note. For a fibration $q : C \to B$, global elements of $p_*(q)$ are sections of q:

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "1_A"'] & & p_*(C) \arrow[dl, "p_*(q)"] \\ & A & \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} B \arrow[rr] \arrow[dr, "1_B"'] & & C \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Models of HoTT: universes

We now assume that there is a notion of 'smallness' for the maps of $\mathcal{E}$ (e.g. given by a bound on the cardinality of fibers).

(4) There is a fibrant object U and a small fibration

$$\pi : \tilde{U} \to U$$

which weakly classifies small fibrations, i.e. for all such $p : B \to A$ there is a pullback

$$\begin{tikzcd} B \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{U} \arrow[d, "\pi"] \\ A \arrow[r] & U \end{tikzcd}$$

Note. We do not ask for uniqueness of the pullback.

Models of HoTT: univalence

(5) The fibration $\pi : \tilde{U} \to U$ is univalent.

In SSet, this holds if and only if for every small fibration $p : B \to A$, the space of squares

$$\begin{tikzcd} B \arrow[r] \arrow[d, "p"'] & \tilde{U} \arrow[d, "\pi"] \\ A \arrow[r] & U \end{tikzcd}$$

such that

$$B \to A \times_U \tilde{U}$$

is a weak equivalence, is contractible.

Question: how can we define examples of models of HoTT?

Quillen model categories

Fix $\mathcal{E}$ with a Quillen model structure $(\mathsf{Weq}, \mathsf{Fib}, \mathsf{Cof})$.

Let $\mathsf{TrivFib} = \mathsf{Weq} \cap \mathsf{Fib}\,,\;\; \mathsf{TrivCof} = \mathsf{Weq} \cap \mathsf{Cof}.$

Question: Is $(\mathcal{E}, \mathsf{Fib})$ a model of HoTT?

Let's look at the axioms for a model of HoTT:

(1) : pullbacks exist and preserve fibrations.

(2) : given by factorization as trivial cofibration followed by a fibration.

The Frobenius property

Lemma. Assume that, for a fibration $p \colon B \to A$, we have

$$\begin{tikzcd} \mathcal{E}/A \arrow[r, shift left, "p^*"] \arrow[r, phantom, "\perp"] & \mathcal{E}/B \arrow[l, shift left, "p_*"] \end{tikzcd}$$

TFAE:

(i) p_* preserves fibrations

(ii) p^* preserves trivial cofibrations.

Definition. A wfs (L, R) is said to have the *Frobenius property* if pullback along R-maps preserves L-maps.

Remark. Assume that $\mathrm{Cof} = \{\text{monomorphisms}\}$. TFAE:

(i) The wfs $(\mathrm{TrivCof}, \mathrm{Fib})$ has the Frobenius property.

(ii) The model structure is right proper, i.e. pullback of weak equivalences along fibrations are weak equivalences.

The fibration extension property

Assume $\mathcal{E} = \mathsf{Psh}(\mathbb{C})$, cofibrations $\subseteq$ monos, fibrations are local (Cisinski).

Lemma. Assume $\pi : \tilde{U} \to U$ classifies small fibrations. Then TFAE:

(i) the universe U is fibrant,

(ii) small fibrations can be extended along trivial cofibrations, i.e.

$$\begin{tikzcd} B \arrow[r, dotted] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d, dotted, "p'"] \\ A \arrow[r, "i"'] & A' \end{tikzcd}$$

(iii) small fibrations can be extended along generating trivial cofibrations.

We call (ii) the **fibration extension property**.

Quillen model categories: the glueing property

Lemma. TFAE:

(i) the fibration $\pi : \tilde{U} \to U$ is univalent,

(ii) weak equivalences between small fibrations can be extended along cofibrations:

$$\begin{tikzcd}
B_1 \arrow[rrr, dotted] \arrow[drr, "w"] \arrow[ddr, "p_1"'] & & & \bullet \arrow[drr, dotted] \arrow[ddr, dotted] & & \\
& & B_2 \arrow[rrr] \arrow[dl, "p_2"] & & & B_2' \arrow[dl, "q_2"] \\
& A \arrow[rrr, "i"'] & & & A' &
\end{tikzcd}$$

We call (ii) the **glueing property** (cf. Coquand et al.)

Example: simplicial sets

Let SSet be the category of simplicial sets.

We consider the model structure for Kan complexes.

Right properness

- via geometric realization (see Hovey, Hirschhorn)
- via minimal fibrations (Joyal and Tierney)

Fibration extension property

- via minimal fibrations and theory of bundles (Joyal)

Glueing property

- Direct proof (Voevodsky)
- Via theory of fiber bundles (Moerdijk)

Issues

Theorem (Bezem, Coquand, Parmann)**.** The right properness of SSet cannot be proved constructively.

A constructive proof is essential for applications in mathematical logic.

How can we fix this?

Coquand's approach

- Switch from simplicial sets to cubical sets
- Work with uniform fibrations. This is useful also to deal with coherence (Swan, Larrea-Schiavon).

Plan:

- alternative presentation of cubical set model
- analysis via the notions of an algebraic weak factorization system.

Goal

For a category $\mathcal{E}$, we want:

(1) to construct an algebraic weak factorization system $(\mathsf{Cof}, \mathsf{TrivFib})$

(2) to construct an algebraic weak factorization system $(\mathsf{TrivCof}, \mathsf{Fib})$

(3) to show that $(\mathsf{TrivCof}, \mathsf{Fib})$ has the Frobenius property.

(4) to prove the glueing property.

(5) to prove the fibration extension property

(6) to show that we have an algebraic model structure.

(1)-(3) this talk, (4)-(6) next talk.

The approach to (1)-(2) is inspired by Cisinski's theory.

Part II: Uniform fibrations and the Frobenius property

Algebraic weak factorization systems

For a weak factorization system, we often ask for

- *functorial factorizations*, i.e. functors (L, R) such that

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "L_f"'] & & B \\ & \bullet \arrow[ur, "R_f"'] & \end{tikzcd}$$

  gives the required factorization.

In an *algebraic weak factorization system*, we also ask that

- L has the structure of a comonad,
- R has the structure of a monad,
- a distributive law between L and R.

Grandis and Tholen (2006), Garner (2009).

Uniform liftings

Fix a category $\mathcal{E}$. Let $u : \mathcal{I} \to \mathcal{E}^{\to}$ be a functor.

Definition. A *right $\mathcal{I}$-map* is a map $p : B \to A$ in $\mathcal{E}$ equipped with

- a function ϕ which assigns a diagonal filler

$$\begin{tikzcd} X_i \arrow[r, "s"] \arrow[d, "u_i"'] & B \arrow[d, "p"] \\ Y_i \arrow[r, "t"'] \arrow[ur, dotted] & A \end{tikzcd}$$

  for $i \in \mathcal{I}$, subject to a uniformity condition:

$$\begin{tikzcd} X_j \arrow[r] \arrow[d, "u_j"'] & X_i \arrow[r, "s"] \arrow[d] & B \arrow[d, "p"] \\ Y_j \arrow[r] \arrow[urr, dotted] & Y_i \arrow[r, "t"'] \arrow[ur, dotted] & A \end{tikzcd}$$

$\mathcal{I}^{\pitchfork}$ = category of right $\mathcal{I}$-maps.

The setting (I)

Let $\mathcal{E}$ be a presheaf category.

We assume a functorial cylinder

$$X \mapsto I \otimes X$$

with endpoint inclusions $\delta^k \otimes X : X \to I \otimes X$ such that

(C1) the cylinder has contractions, $\varepsilon_X : I \otimes X \to X$

(C2) the cylinder has connections, $c^k_X : I \otimes I \otimes X \to I \otimes X$

(C3) $I \otimes (-)$ has a right adjoint

(C4) $I \otimes (-) : \mathcal{E} \to \mathcal{E}$ preserves pullback squares

(C5) the endpoint inclusions $\delta^k \otimes X : X \to I \otimes X$ are cartesian.

Examples: SSet , CSet.

The setting (II)

We also fix a full subcategory

$$\mathcal{M} \hookrightarrow \mathcal{E}^{\to}_{\mathsf{cart}}$$

of monomorphisms such that:

(M1) the unique map $\perp_X : 0 \to X$ is in $\mathcal{M}$, for every $X \in \mathcal{E}$

(M2) $\mathcal{M}$ is closed under pullbacks

(M3) $\mathcal{M}$ is closed under pushout product with the endpoint inclusions.

Examples: $\mathcal{M} =$ all monomorphisms, in SSet or CSet.

Uniform trivial fibrations

Fix $\mathcal{E}$, $I \otimes (-)$, $\mathcal{M}$ as above.

Write $u : \mathcal{M} \to \mathcal{E}^{\to}$ for the inclusion.

Definition. A *uniform trivial fibration* is a right $\mathcal{M}$-map, i.e. a map $f : B \to A$ together with a function which assigns fillers

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "i"'] & B \arrow[d, "f"] \\ Y \arrow[r, "t"'] \arrow[ur, dotted] & A \end{tikzcd}$$

where $i : X \to Y$ is a monomorphism in $\mathcal{M}$, subject to uniformity.

$$\mathrm{TrivFib} = \mathcal{M}^{\pitchfork} = \text{category of uniform trivial fibrations.}$$

Cylinder inclusions

For a monomorphism $i : X \to Y$ in $\mathcal{M}$, we have the **pushout product**

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "\delta_k \otimes X"'] & Y \arrow[d] \arrow[ddr, bend left, "\delta^k \otimes Y"] & \\
I \otimes X \arrow[r] \arrow[drr, bend right, "I \otimes i"'] & \bullet \arrow[dr, dotted, "\delta^k \hat{\otimes} i"] & \\
 & & I \otimes Y
\end{tikzcd}$$

We get a subcategory $\mathrm{Cyl} \subseteq \mathcal{E}^{\to}$ with objects the “cylinder inclusions”

$$\delta^k \hat{\otimes} i \, : \, \underbrace{(I \otimes X) \cup (\{k\} \otimes Y)}_{\bullet} \to I \otimes Y$$

Uniform fibrations

Definition. A *uniform fibration* is a right Cyl-map, i.e. a map $p : B \to A$ together with a function which assigns fillers

$$\begin{tikzcd}
(I \otimes X) \cup (\{k\} \otimes Y) \arrow[r] \arrow[d, "\delta^k \hat{\otimes} i"'] & B \arrow[d, "p"] \\
I \otimes Y \arrow[r] \arrow[ur, dotted] & A
\end{tikzcd}$$

where $i : X \to Y$ is a monomorphism in $\mathcal{M}$, subject to uniformity.

$$\mathsf{Fib} = \mathsf{Cyl}^{\pitchfork} = \text{category of uniform fibrations.}$$

Theorem*. *A map is a (trivial) fibration in the usual sense if and only if it can be equipped with the structure of a uniform (trivial) fibration.*

The algebraic weak factorization systems

Theorem. $\mathcal{E}$ admits two cofibrantly-generated algebraic weak factorization systems:

1. $(\mathsf{Cof}, \mathsf{TrivFib})$
2. $(\mathsf{TrivCof}, \mathsf{Fib})$.

Proof.

- Use Garner's algebraic small object argument
- For this, isolate a small category $\mathcal{I}$ such that

$$\mathcal{I}^{\pitchfork} = \mathcal{M}^{\pitchfork} \quad (= \mathsf{TrivFib})$$

- E.g. $\mathcal{I} = \{$monomorphisms in $\mathcal{M}$ with representable codomain$\}$.

Note. Algebraic aspect is essential to work constructively.

The Frobenius property

We want to show that $(\mathrm{TrivCof}, \mathrm{Fib})$ has the Frobenius property.

For simplicity, we work in the non-algebraic setting.

Recall that we have a class of maps Cyl such that $\mathrm{Cyl}^{\pitchfork} = \mathrm{Fib}$

To show:

- for $p : B \to A$ in Fib, pullback

  $$p^* : \mathcal{E}/A \to \mathcal{E}/B$$

  preserves trivial cofibrations, i.e. for all

  $$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ B \arrow[r, "p"'] & A \end{tikzcd}$$

  we have $f \in \mathrm{TrivCof} \Rightarrow g \in \mathrm{TrivCof}$.

Outline of the proof

Define the class SHeq of strong homotopy equivalences

Step 1

- characterize strong homotopy equivalences as retracts

Step 2

- Show $\mathsf{SHeq} \cap \mathcal{M} \subseteq \overline{\mathsf{Cyl}}$
- Show $\mathsf{Cyl} \subseteq \mathsf{SHeq} \cap \mathcal{M}$

Step 3

- Prove the Frobenius property for $\mathsf{SHeq} \cap \mathcal{M}$

Strong homotopy equivalences

Definition. A map $f : X \to A$ is a *strong left homotopy equivalence* if there exist

$$g : A \to X \,, \qquad \phi : g \circ f \sim 1_X \quad \psi : f \circ g \sim 1_A$$

such that

$$\begin{tikzcd}
I \otimes X \arrow[r, "I \otimes f"] \arrow[d, "\phi"'] & I \otimes A \arrow[d, "\psi"] \\
X \arrow[r, "f"'] & A
\end{tikzcd}$$

Example. The endpoint inclusion $\delta^0 \otimes X : X \to I \otimes X$.

There is a dual notion of strong right homotopy equivalence.

Step 1: a characterisation

Lemma. *A map $f : X \to A$ is a strong left homotopy equivalence if and only if the canonical square*

$$\begin{tikzcd} X \arrow[r, "\delta^1 \otimes X"] \arrow[d, "f"'] & I \otimes X \arrow[r, "\iota_1"] & (I \otimes X) +_X A \arrow[d, "\delta^0 \hat{\otimes} f"] \\ A \arrow[rr, "\delta^1 \otimes A"'] & & I \otimes A \end{tikzcd}$$

exhibits f as a retract of $\delta^0 \hat{\otimes} f$, i.e. we have

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & (I \otimes X) +_X A \arrow[r, "s"] \arrow[d, "\delta^0 \hat{\otimes} f"] & X \arrow[d, "f"] \\ A \arrow[r, "\delta^1 \otimes A"'] & I \otimes A \arrow[r, "t"'] & A \end{tikzcd}$$

where the horizontal composites are identities.

Step 2: a lemma

Lemma. *We have*

(i) $\mathsf{SHeq} \cap \mathcal{M} \subseteq \overline{\mathsf{Cyl}}$

(ii) $\mathsf{Cyl} \subseteq \mathsf{SHeq} \cap \mathcal{M}$

Proof.

(i) Let $f \in \mathsf{SHeq} \cap \mathcal{M}$.

Since $f \in \mathsf{SHeq}$, by Step 1, we have that f is a retract of, say, $\delta^0 \hat{\otimes} f$.

Since $f \in \mathcal{M}$, we have $\delta^0 \hat{\otimes} f \in \mathsf{Cyl}$.

(ii) Each $\delta^0 \hat{\otimes} f \in \mathsf{Cyl}$ is both in SHeq and in $\mathcal{M}$.

Step 3: end of the proof

Theorem. *The weak factorization system* $(\mathrm{TrivCof}, \mathrm{Fib})$ *has the Frobenius property.*

Proof. We need to show that for every pullback

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ B \arrow[r, "p"'] & A \end{tikzcd}$$

where $p \in \mathrm{Fib}$, we have

$$f \in \mathrm{TrivCof} \Rightarrow g \in \mathrm{TrivCof}$$

But by Step 2, it suffices to show

$$f \in \mathrm{SHeq} \cap \mathcal{M} \Rightarrow g \in \mathrm{SHeq} \cap \mathcal{M}$$

Let $f \in \mathsf{SHeq} \cap \mathcal{M}$. To show: $g = p^*(f) \in \mathsf{SHeq} \cap \mathcal{M}$.

By Step 1 and some diagram-chasing, we need

$$
\begin{tikzcd}
B \arrow[r, "\delta^0 \otimes B"] \arrow[d, "p"'] & I \otimes B \arrow[r, dotted] \arrow[d] & B \arrow[d, "p"] \\
A \arrow[r, "\delta^0 \otimes A"'] & I \otimes A \arrow[r, "t"'] & A
\end{tikzcd}
$$

Here t is part of the data making f into a retract of $\delta^1 \hat{\otimes} f$.

Such a map is given by a diagonal filler:

$$
\begin{tikzcd}
B \arrow[rr, "1_B"] \arrow[d, "\mathsf{Cyl} \ni \delta^0 \otimes B"'] & & B \arrow[d, "p \in \mathsf{Fib}"] \\
I \otimes B \arrow[r, "I \otimes p"'] \arrow[urr, dotted] & I \otimes A \arrow[r, "t"'] & A
\end{tikzcd}
$$

Summary

Done:

(1) algebraic weak factorization system $(\mathrm{Cof}, \mathrm{TrivFib})$

(2) algebraic weak factorization system $(\mathrm{TrivCof}, \mathrm{Fib})$

(3) $(\mathrm{TrivCof}, \mathrm{Fib})$ has the Frobenius property.

Examples

- CSet
- SSet, so get new proof that SSet is right proper.

Still to do:

(4) To prove the glueing property

(5) To prove the fibration extension property

(6) To show that we have an algebraic model structure.