Homotopy Type Theory and Algebraic Model Structures (I)

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Topologie Algébrique et Applications

Paris, 2nd December 2016

Plan of the talks

Goal

analysis of the cubical set model of Homotopy Type Theory

By-products

- general method to obtain right proper algebraic model structures,
- a new proof of model structure for Kan complexes and its right properness, avoiding minimal fibrations.

Outline of this talk

Part I: Homotopy Type Theory

- Models of HoTT
- Quillen model structures
- Some issues

Part II: Uniform fibrations and the Frobenius property

- Algebraic weak factorization systems
- Uniform fibrations
- The Frobenius property

References

- C. Cohen, T. Coquand, S. Huber and A. Mörtberg Cubical Type Theory: a constructive interpretation of the univalence axiom arXiv, 2016.
- N. Gambino and C. Sattler Frobenius condition, right properness, and uniform fibrations arXiv, 2016.

Part I: Homotopy Type Theory

Homotopy Type Theory

 $\mathsf{HoTT} = \mathsf{Martin-L\"of's type theory} + \mathsf{Voevodsky's univalence axiom}$

Key ingredients:

- (1) substitution operation,
- (2) identity types,
- (3) Π-types,
- (4) a type universe,
- (5) univalence axiom.

We give a category-theoretic account of these.

Models of HoTT

Definition. A model of homotopy type theory consists of

- a category $\mathcal E$ with a terminal object 1,
- > a class Fib of maps, called *fibrations*,

subject to axioms (1)-(5).

Idea: the sequent

 $x: A \vdash B(x): type$

is interpreted as a fibration

$$B \\ \downarrow_p \\ A$$

Warning: issues of coherence will be ignored.

Models of HoTT: substitution

(1) Pullbacks of fibrations exist and are again fibrations.

So for every map $\sigma: \mathcal{A}' \to \mathcal{A}$ we have

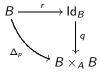
$$\operatorname{Fib}/A \xrightarrow{\sigma^*} \operatorname{Fib}/A'$$

Diagrammatically:



Models of HoTT: identity types

(2) For every fibration $p: B \rightarrow A$, there is a factorization



where $r \in {}^{\pitchfork}\mathsf{Fib}$ and $q \in \mathsf{Fib}$.

Models of HoTT: Π-types

(3) If $p: B \to A$ is a fibration, pullback along p has a right adjoint

$$\operatorname{Fib}/B \xrightarrow{p_*} \operatorname{Fib}/A$$

We call this the *pushforward* along *p*.

Note. For a fibration $q: C \rightarrow B$, global elements of $p_*(q)$ are sections of q:



Models of HoTT: universes

We now assume that there is a notion of 'smallness' for the maps of \mathcal{E} (e.g. given by a bound on the cardinality of fibers).

(4) There is a fibrant object U and a small fibration

$$\pi: \tilde{U} \to U$$

which weakly classifies small fibrations, i.e. for all such $p: B \rightarrow A$ there is a pullback



Note. We do not ask for uniqueness of the pullback.

Models of HoTT: univalence

(5) The fibration $\pi: \tilde{U} \to U$ is univalent.

In SSet, this holds if and only if for every small fibration $p: B \rightarrow A$, the space of squares



such that

$$B \to A \times_U \tilde{U}$$

is a weak equivalence, is contractible.

Question: how can we define examples of models of HoTT?

Quillen model categories

Fix \mathcal{E} with a Quillen model structure (Weq, Fib, Cof). Let TrivFib = Weg \cap Fib, TrivCof = Weg \cap Cof.

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Question: Is (\mathcal{E}, Fib) a model of HoTT?
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Let's look at the axioms for a model of HoTT:

- (1) : pullbacks exist and preserve fibrations.
- (2) : given by factorization as trivial cofibration followed by a fibration.

The Frobenius property

Lemma. Assume that, for a fibration $p: B \rightarrow A$, we have

$$\mathcal{E}/A \xrightarrow{p^*} \mathcal{E}/B$$

TFAE:

(i) p_* preserves fibrations

(ii) p^* preserves trivial cofibrations.

Definition. A wfs (L, R) is said to have the *Frobenius property* if pullback along R-maps preserves L-maps.

Remark. Assume that Cof = {monomorphisms}. TFAE:

- (i) The wfs (TrivCof, Fib) has the Frobenius property.
- (ii) The model structure is right proper, i.e. pullback of weak equivalences along fibrations are weak equivalences.

The fibration extension property

Assume $\mathcal{E} = \mathsf{Psh}(\mathbb{C})$, cofibrations \subseteq monos, fibrations are local (Cisinski).

Lemma. Assume $\pi: \tilde{U} \to U$ classifies small fibrations. Then TFAE:

(i) the universe U is fibrant,

(ii) small fibrations can be extended along trivial cofibrations, i.e.



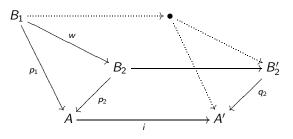
(iii) small fibrations can be extended along generating trivial cofibrations.

We call (ii) the fibration extension property.

Quillen model categories: the glueing property

Lemma. TFAE:

- (i) the fibration $\pi: \widetilde{U}
 ightarrow U$ is univalent,
- (ii) weak equivalences between small fibrations can be extended along cofibrations:



We call (ii) the glueing property (cf. Coquand et al.)

Example: simplicial sets

Let SSet be the category of simplicial sets.

We consider the model structure for Kan complexes.

Right properness

- via geometric realization (see Hovey, Hirschhorn)
- via minimal fibrations (Joyal and Tierney)

Fibration extension property

via minimal fibrations and theory of bundles (Joyal)

Glueing property

- Direct proof (Voevodsky)
- Via theory of fiber bundles (Moerdijk)

Issues

Theorem (Bezem, Coquand, Parmann). The right properness of SSet cannot be proved constructively.

A constructive proof is essential for applications in mathematical logic.

How can we fix this?

Coquand's approach

- Switch from simplicial sets to cubical sets
- ▶ Work with uniform fibrations. This is useful also to deal with coherence (Swan, Larrea-Schiavon).

Plan:

- alternative presentation of cubical set model
- ▶ analysis via the notions of an algebraic weak factorization system.

Goal

For a category ${\mathcal E},$ we want:

- (1) to construct an algebraic weak factorization system (Cof, TrivFib)
- (2) to construct an algebraic weak factorization system (TrivCof, Fib)
- (3) to show that (TrivCof, Fib) has the Frobenius property.
- (4) to prove the glueing property.
- (5) to prove the fibration extension property
- (6) to show that we have an algebraic model structure.

(1)-(3) this talk, (4)-(6) next talk.

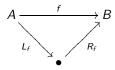
The approach to (1)-(2) is inspired by Cisinski's theory.

Part II: Uniform fibrations and the Frobenius property

Algebraic weak factorization systems

For a weak factorization system, we often ask for

• functorial factorizations, i.e. functors (L, R) such that



gives the required factorization.

In an algebraic weak factorization system, we also ask that

- L has the structure of a comonad,
- R has the structure of a monad,
- ▶ a distributive law between *L* and *R*.

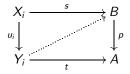
Grandis and Tholen (2006), Garner (2009).

Uniform liftings

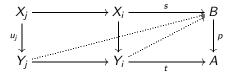
Fix a category \mathcal{E} . Let $u: \mathcal{I} \to \mathcal{E}^{\to}$ be a functor.

Definition. A right \mathcal{I} -map is a map $p: B \to A$ in \mathcal{E} equipped with

 \blacktriangleright a function ϕ which assigns a diagonal filler



for $i \in \mathcal{I}$, subject to a uniformity condition:



 $\mathcal{I}^{\pitchfork} = \mathsf{category} \text{ of right } \mathcal{I}\text{-maps.}$

The setting (I)

Let $\ensuremath{\mathcal{E}}$ be a presheaf category.

We assume a functorial cylinder

$$X \mapsto I \otimes X$$

with endpoint inclusions $\delta^k \otimes X : X \to I \otimes X$ such that

(C1) the cylinder has contractions, $\varepsilon_X : I \otimes X \to X$ (C2) the cylinder has connections, $c_X^k : I \otimes I \otimes X \to I \otimes X$ (C3) $I \otimes (-)$ has a right adjoint (C4) $I \otimes (-) : \mathcal{E} \to \mathcal{E}$ preserves pullback squares (C5) the endpoint inclusions $\delta^k \otimes X : X \to I \otimes X$ are cartesian.

Examples: SSet, CSet.

The setting (II)

We also fix a full subcategory

$$\mathcal{M} \hookrightarrow \mathcal{E}_{\mathsf{cart}}^{\rightarrow}$$

of monomorphisms such that:

(M1) the unique map $\bot_X : 0 \to X$ is in \mathcal{M} , for every $X \in \mathcal{E}$

(M2) \mathcal{M} is closed under pullbacks

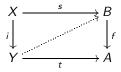
(M3) \mathcal{M} is closed under pushout product with the endpoint inclusions.

Examples: $\mathcal{M} = \text{all monomorphisms, in SSet or CSet.}$

Uniform trivial fibrations

Fix \mathcal{E} , $I \otimes (-)$, \mathcal{M} as above. Write $u : \mathcal{M} \to \mathcal{E}^{\to}$ for the inclusion.

Definition. A *uniform trivial fibration* is a right M-map, i.e. a map $f: B \to A$ together with a function which assigns fillers

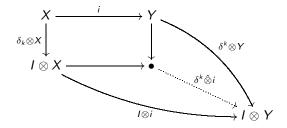


where $i: X \to Y$ is a monomorphism in \mathcal{M} , subject to uniformity.

 $\mathsf{TrivFib} = \mathcal{M}^{\pitchfork} = \mathsf{category} \text{ of uniform trivial fibrations.}$

Cylinder inclusions

For a monomorphism $i: X \to Y$ in \mathcal{M} , we have the **pushout product**

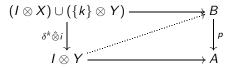


We get a subcategory $\mathsf{Cyl}\subseteq \mathcal{E}^{\rightarrow}$ with objects the "cylinder inclusions"

$$\delta^k \hat{\otimes} i : \underbrace{(I \otimes X) \cup (\{k\} \otimes Y)}_{\bullet} \to I \otimes Y$$

Uniform fibrations

Definition. A *uniform fibration* is a right Cyl-map, i.e. a map $p: B \rightarrow A$ together with a function which assigns fillers



where $i: X \to Y$ is a monomorphism in \mathcal{M} , subject to uniformity.

 $Fib = Cyl^{\uparrow\uparrow} = category of uniform fibrations.$

Theorem^{*}. A map is a (trivial) fibration in the usual sense if and only if it can be equipped with the structure of a uniform (trivial) fibration.

The algebraic weak factorization systems

Theorem. $\ensuremath{\mathcal{E}}$ admits two cofibrantly-generated algebraic weak factorization systems:

- 1. (Cof, TrivFib)
- 2. (TrivCof, Fib).

Proof.

- Use Garner's algebraic small object argument
- \blacktriangleright For this, isolate a small category ${\mathcal I}$ such that

$$\mathcal{I}^{\pitchfork} = \mathcal{M}^{\pitchfork} \quad (= \mathsf{TrivFib})$$

• E.g. $\mathcal{I} = \{\text{monomorphisms in } \mathcal{M} \text{ with representable codomain} \}.$

Note. Algebraic aspect is essential to work constructively.

The Frobenius property

We want to show that (TrivCof, Fib) has the Frobenius property. For simplicity, we work in the non-algebraic setting.

Recall that we have a class of maps Cyl such that ${\rm Cyl}^{\pitchfork}={\rm Fib}$ To show:

• for $p: B \rightarrow A$ in Fib, pullback

 $p^*: \mathcal{E}/A \to \mathcal{E}/B$

preserves trivial cofibrations, i.e. for all

$$\begin{array}{c} Y \longrightarrow X \\ g \downarrow^{-} & \downarrow^{f} \\ B \longrightarrow A \end{array}$$

we have $f \in \text{TrivCof} \Rightarrow g \in \text{TrivCof}$.

Outline of the proof

Define the class SHeq of strong homotopy equivalences

Step 1

characterize strong homotopy equivalences as retracts

Step 2

- $\blacktriangleright \text{ Show SHeq} \cap \mathcal{M} \subseteq \overline{\mathsf{Cyl}}$
- $\blacktriangleright \ \ \mathsf{Show} \ \ \mathsf{Cyl} \subseteq \mathsf{SHeq} \cap \mathcal{M}$

Step 3

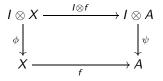
 \blacktriangleright Prove the Frobenius property for $\mathsf{SHeq}\cap\mathcal{M}$

Strong homotopy equivalences

Definition. A map $f: X \to A$ is a strong left homotopy equivalence if there exist

 $g: A \to X, \qquad \phi: g \circ f \sim 1_X \quad \psi: f \circ g \sim 1_A$

such that

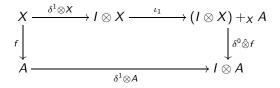


Example. The endpoint inclusion $\delta^0 \otimes X : X \to I \otimes X$.

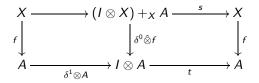
There is a dual notion of strong right homotopy equivalence.

Step 1: a characterisation

Lemma. A map $f : X \to A$ is a strong left homotopy equivalence if and only if the canonical square



exhibits f as a retract of $\delta^0 \hat{\otimes} f$, i.e. we have



where the horizontal composites are identities.

Step 2: a lemma

Lemma. We have

- (i) $\mathsf{SHeq} \cap \mathcal{M} \subseteq \overline{\mathsf{Cyl}}$
- (ii) $\mathsf{Cyl} \subseteq \mathsf{SHeq} \cap \mathcal{M}$

Proof.

(i) Let $f \in SHeq \cap M$.

Since $f \in SHeq$, by Step 1, we have that f is a retract of, say, $\delta^0 \hat{\otimes} f$. Since $f \in \mathcal{M}$, we have $\delta^0 \hat{\otimes} f \in Cyl$.

(ii) Each $\delta^0 \hat{\otimes} f \in Cyl$ is both in SHeq and in \mathcal{M} .

Step 3: end of the proof

Theorem. The weak factorization system (TrivCof, Fib) has the Frobenius property.

Proof. We need to show that for every pullback



where $p \in Fib$, we have

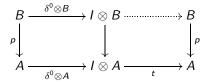
$$f \in \mathsf{TrivCof} \Rightarrow g \in \mathsf{TrivCof}$$

But by Step 2, it suffices to show

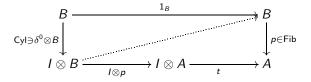
$$f\in\mathsf{SHeq}\cap\mathcal{M}\Rightarrow g\in\mathsf{SHeq}\cap\mathcal{M}$$

Let $f \in \mathsf{SHeq} \cap \mathcal{M}$. To show: $g = p^*(f) \in \mathsf{SHeq} \cap \mathcal{M}$.

By Step 1 and some diagram-chasing, we need



Here t is part of the data making f into a retract of $\delta^1 \hat{\otimes} f$. Such a map is given by a diagonal filler:



Summary

Done:

(1) algebraic weak factorization system (Cof, TrivFib)

(2) algebraic weak factorization system (TrivCof, Fib)

(3) (TrivCof, Fib) has the Frobenius property.

Examples

- CSet
- SSet, so get new proof that SSet is right proper.

Still to do:

- (4) To prove the glueing property
- (5) To prove the fibration extension property
- (6) To show that we have an algebraic model structure.