Constructive Mathematics in Constructive Set Theory

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Classical vs Constructive Mathematics

	AC	EM	Pow
ZFC	✓	✓	✓
ZF	✗	✓	✓
IZF	✗	✗	✓
Topos Theory	✗	✗	✓
Constructive Set Theory	✗	✗	✗
Constructive Type Theory	✗	✗	✗

Constructive topology?

Problem

- Can we develop topology in Constructive Set Theory?

Issues

- Sometimes AC is essential. E.g.

$$\text{Tychonoff's Theorem} \iff \text{Axiom of Choice.}$$

- Use of EM and Pow is widespread in classical topology.
- Classically equivalent structures become distinct. E.g.

$$\text{Dedekind reals} \neq \text{Cauchy reals.}$$

Some developments

Pointfree topology (Banaschewski, Isbell, Johnstone, Vickers, ...)

- Traditionally developed in ZF or Topos Theory
- Focus on frames and locales

Formal Topology (Martin-Löf, Sambin, Coquand, Schuster, Palmgren, ...)

- Traditionally developed in Constructive Type Theory
- Focus on formal topologies

Formal Topology in CST (Aczel, Curi, Palmgren ...)

- Work in CZF, CZF^+ or even fragments of CZF
- Focus on both frames and formal topologies

Outline

Part I: The basic notions

- Set-generated frames
- Formal topologies

Part II: Further topics

- Covering systems
- Inductively defined formal topologies
- The fundamental adjunction

Part I

The basic notions

From topological spaces to frames

Let $(X, \mathcal{O}(X))$ be a topological space. The set $\mathcal{O}(X)$ is

- a partial order
$$U \leq V =_{\text{def}} U \subseteq V \, ,$$
- a complete join-semilattice
$$\bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i \, ,$$
- a meet-semilattice:
$$U \wedge V =_{\text{def}} U \cap V \, .$$

Furthermore, it satisfies the distributivity law

$$U \wedge \bigvee_{i \in I} V_i = \bigvee_{i \in I} (U \wedge V_i)$$

Frames

Definition. A **frame** is a partially ordered set $(A, \leq)$ having arbitrary joins and binary meets which satisfy the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge x \mid x \in S\}$$

for every $a \in A$ and $S \subseteq A$.

Note. Every frame is a complete lattice, since

$$\bigwedge S =_{\text{def}} \bigvee \{a \in A \mid (\forall x \in S)\, a \leq x\}\,.$$

Examples. $\mathcal{P}(X)$ is a frame.

Pointfree topology

Key idea

- Replace topological spaces by frames
- Work with frames as ‘generalized spaces’.

Fundamental adjunction

$$\mathcal{O}\colon \mathbf{Top} \longleftrightarrow \mathbf{Frm}^{\mathrm{op}} \colon \mathrm{Pt}$$

where

- $\mathbf{Top}$ = category of topological spaces and continuous maps,
- $\mathbf{Frm}$ = category of frames and frame homomorphisms.

Problems for Constructive Set Theory

If we try to represent this in CZF we run into problems, e.g.

- $\mathcal{O}(X)$ is not a frame in CZF, since in general it is not a set.
- $\mathcal{P}(X)$ is not ...

Idea

- We allow frames to be classes.
- We add data to have arbitrary meets and top element.

Set-generated frames

New Definition. A **frame** is a partially ordered class $(A, \leq)$ with joins for all $S \in \mathcal{P}(A)$, a top element and binary meets satisfying the distributivity law.

Definition. A **set-generated frame** is a frame equipped with a **generating set**, i.e. a set G such that

- For all $a \in A$, the class $G_a =_{\mathrm{def}} \{x \in G \mid x \leq a\}$ is a set.
- For all $a \in A$, we have $a = \bigvee G_a$.

Observation. In a set-generated frame, we can define the meet of $S \in \mathcal{P}(A)$ by

$$\bigwedge S =_{\mathrm{def}} \bigvee \{a \in G \mid (\forall x \in S)\, x \leq a\}$$

Examples

- Let X be a set. The class $\mathcal{P}(X)$ is a set-generated frame. A generating set is $\{\{x\} \mid x \in X\}$.
- Let $(X, \leq)$ be a poset. A **lower subset** of X is a subset $U \subseteq X$ such that

  $$U = {\downarrow} U$$

  where

  $$ {\downarrow} U =_{\mathrm{def}} \{x \in X \mid (\exists u \in U)\, x \leq u\}$$

  The class $\mathcal{L}(X)$ of lower subsets is a set-generated frame. The generating set is $\{{\downarrow}\{x\} \mid x \in X\}$.

It is convenient to have an alternative way of working with set-generated frames.

Formal topologies

Definition. A **formal topology** consists of a poset $(S, \leq)$ equipped with a **cover relation**, i.e. a relation

$$a \lhd U \qquad (\text{for } a \in S\,,\ U \in \mathcal{P}(S))$$

such that

(1) if $a \in U$ then $a \lhd U$,

(2) if $a \leq b$ and $b \lhd U$ then $a \lhd U$,

(3) if $a \lhd U$ and $U \lhd V$ then $a \lhd V$,

(4) if $a \lhd U$ and $a \lhd V$ then $a \lhd U \downarrow V$,

(5) for every $U \in \mathcal{P}(S)$, the class $\{x \in S \mid x \lhd U\}$ is a set,

where

$$U \lhd V \quad =_{\text{def}} \quad (\forall x \in U)\, x \lhd V\,,$$

$$U \downarrow V \quad =_{\text{def}} \quad {\downarrow} U \cap {\downarrow} V\,.$$

Formal topologies vs set-generated frames

Proposition.

1. For a set-generated frame $(A, \leq, \bigvee, \wedge, \top, G)$, we can define a cover relation on $(G, \leq)$ by letting
$$a \lhd U \iff a \leq \bigvee U\,.$$
2. For a formal topology $(S, \leq, \lhd)$, the class of the subsets $U \subseteq S$ such that
$$U = \{x \in S \mid x \lhd U\}$$
has the structure of a set-generated frame.

Note. This result extends to an equivalence of categories.

Points

Let $(S, \leq, \lhd)$ be a formal topology.

Definition. ** A **point of S is a subset $\alpha \subseteq S$ such that, letting

$$\alpha \Vdash a =_{\text{def}} a \in \alpha\,,$$

we have that

1. α is inhabited
2. If $\alpha \Vdash a$ and $a \leq b$ then $\alpha \Vdash b$
3. If $\alpha \Vdash a_1$, $\alpha \Vdash a_2$ then there is $a \leq a_1, a_2$ such that $\alpha \Vdash a$
4. If $\alpha \Vdash a$ and $a \lhd U$ then there is $x \in U$ such that $\alpha \Vdash x$.

Note. The points of S form a (large) topological space, $\mathrm{Pt}(S)$.

Example: the formal Dedekind reals

Define a formal topology $(\mathcal{R}, \leq, \lhd)$ as follows:

- $\mathcal{R} =_{\text{def}} \{(p,q) \mid p \in \mathbb{Q} \cup \{-\infty\}, q \in \mathbb{Q} \cup \{+\infty\}, p < q\}$
- $(p,q) \leq (p',q')$ iff $p' \leq p$ and $q \leq q'$.
- The cover relation is defined inductively by the rules

$$\frac{(p,q) \in U}{(p,q) \lhd U} \qquad \frac{(p,q) \leq (r,s) \quad (r,s) \lhd U}{(p,q) \lhd U}$$

$$\frac{(p,q') \lhd U \quad (p',q) \lhd U}{(p,q) \lhd U} \quad \text{for } p \leq p' \leq q' \leq q$$

$$\frac{\big(\forall (p',q') < (p,q)\big)\, (p',q') \lhd U}{(p,q) \lhd U}$$

Proposition. The space $\mathrm{Pt}(\mathcal{R})$ is homeomorphic to $\mathbb{R}$.

Example: the formal Cantor space

Define a formal topology $(\mathcal{C}, \leq, \lhd)$ as follows:

- $\mathcal{C} =_{\text{def}}$ set of finite sequences of 0's and 1's.
- For $p, q \in \mathcal{C}$, let

$$p \leq q \text{ iff } q \text{ is an initial segment of } p$$

- The cover relation is defined inductively by the rules

$$\frac{p \in U}{p \lhd U} \qquad \frac{p \leq q \qquad q \lhd U}{p \lhd U} \qquad \frac{p \cdot 0 \lhd U \qquad p \cdot 1 \lhd U}{p \lhd U}$$

Proposition. The space $\mathrm{Pt}(\mathcal{C})$ is homeomorphic to $2^{\mathbb{N}}$.

Example: the double negation formal topology

Consider $1 =_{\mathrm{def}} \{0\}$ as a discrete poset and let $\Omega =_{\mathrm{def}} \mathcal{P}(1)$

For $a \in 1$ and $U \in \Omega$ define

$$a \lhd U =_{\mathrm{def}} \neg\neg a \in U\,.$$

To check:

1. If $a \in U$ then $\neg\neg a \in U$
2. If $a = b$ and $\neg\neg b \in U$ then $\neg\neg a \in U$
3. If $\neg\neg a \in U$ and $(\forall x \in U)\,\neg\neg x \in V$ then $\neg\neg a \in V$
4. If $\neg\neg a \in U$ and $\neg\neg a \in V$ then $\neg\neg a \in U \cap V$
5. For every $U \in \Omega$, the class $\{x \in 1 \mid \neg\neg x \in U\}$ is a set.

Part II

Further topics

Covering systems

Let $(S, \leq)$ be a poset.

Definition. A **covering system** on $(S, \leq)$ is a family of sets

$$(\, \mathrm{Cov}(a) \mid a \in S \,)$$

such that

1. if $P \in \mathrm{Cov}(a)$ then $P \subseteq {\downarrow} a$,
2. if $P \in \mathrm{Cov}(a)$ and $b \leq a$, then there is $Q \in \mathrm{Cov}(b)$ such that

$$(\forall y \in Q)(\exists x \in P)\, y \leq x\,.$$

Note. Compare with the notion of a Grothendieck coverage.

Inductively defined formal topologies

Let $(\,\mathrm{Cov}(a) \mid a \in S\,)$ be a covering system on $(S, \leq)$.

We define inductively a cover relation on $(S, \leq)$ by the rules

$$\frac{a \in U}{a \lhd U} \qquad \frac{a \leq b \quad b \lhd U}{a \lhd U} \qquad \frac{P \lhd U}{a \lhd U} \;\text{ for } P \in \mathrm{Cov}(a)$$

Proposition. $(S, \leq, \lhd)$ is a formal topology.

Examples.

- The formal Dedekind reals
- The formal Cantor space

$$U \in \mathrm{Cov}(p) \iff U = \{p \cdot 0\,, p \cdot 1\}$$

A characterization

Theorem (Aczel). A formal topology $(S, \leq, \lhd)$ is inductively defined if and only if it is **set-presented**, i.e. there exists

$$R \colon S \to \mathcal{P}(S)$$

such that

$$a \lhd U \Leftrightarrow (\exists V \in R(a)) V \subseteq U$$

Proof. Application of the Set Compactness Theorem.

Theorem. The double-negation formal topology is not set-presented.

The fundamental adjunction

Classically, there is an adjunction

$$\mathcal{O}\colon \mathbf{Top} \longleftrightarrow \mathbf{Frm}^{\mathrm{op}} \colon \mathrm{Pt}$$

Peter Aczel has obtained a version of this adjunction in CZF

$$\mathcal{O}\colon \mathbf{Top}_1 \longleftrightarrow \mathbf{Frm}_1^{\mathrm{op}} \colon \mathrm{Pt}$$

where

- $\mathbf{Top}_1$ is equivalent to $\mathbf{Top}$ in IZF
- $\mathbf{Frm}_1$ is equivalent to $\mathbf{Frm}$ in IZF

The proof of this result involves subtle size conditions.

References

Pointfree topology

- P. T. Johnstone, Stone Spaces, 1982

Formal topology

- G. Sambin, Intuitionistic formal spaces, 1987
- P. Aczel, Aspects of general topology in CST, 2006