Homotopy-theoretic aspects of Martin-Löf type theory

Nicola Gambino

University of Palermo

visiting

The University of Manchester

Logic Colloquium

Paris – July 30th, 2010

Background

Identity types: for a type A and $a, b \in A$, we have a new type

$$\mathrm{Id}_A(a, b)$$

Idea:

$$p \in \mathrm{Id}_A(a, b) \;\Leftrightarrow\; \text{“}p \text{ is a proof that } a \text{ equals } b\text{”}$$

Key discovery (Hofmann and Streicher, 1995)**:**

$$p\,, q \in \mathrm{Id}_A(a, b) \;\not\Rightarrow\; p = q$$

Question:

- What is the combinatorics of identity types?

Recent advances

Models

- Awodey and Warren (2007), Warren (2008)
- van den Berg and Garner (2010)

Identity types and homotopy theory

- Gambino and Garner (2008)
- Awodey, Hofstra and Warren (2009)

Identity types and higher-dimensional categories

- van den Berg and Garner (2008)
- Lumsdaine (2008)

Voevodsky's work

- Homotopy λ-calculus (2006)
- Univalent models (2010)

Overview

Part I

- Identity types

Part II

- The identity type weak factorization system

Part III

- Weak ω-groupoids

Part I

Identity types

Martin-Löf type theory (I)

Dependent types:

$$x \in A \vdash B(x) \in \mathrm{Type}$$

Key ideas:

- Propositions-as-types
- Theory of inductive definitions
- Computer implementation

Forms of type:

$$0\,, \quad 1\,, \quad \mathbb{N}\,, \quad A \times B\,, \quad A \Rightarrow B\,, \quad A + B\,,$$

$$\mathrm{Id}_A(a,b)\,, \quad \textstyle\prod_{x \in A} B(x)\,, \quad \textstyle\sum_{x \in A} B(x)\,, \quad \ldots$$

We will only need the rules for identity types.

Martin-Löf type theory (II)

Judgements

$$A \in \mathrm{Type}\,, \qquad a \in A\,, \qquad A = B \in \mathrm{Type}\,, \qquad a = b \in A\,.$$

Hypothetical judgements

$$\Gamma \vdash J$$

where $\Gamma = (x_1 \in A_1\,, \ldots\,, x_n \in A_n)$.

Deduction rules

$$\frac{\Gamma_1 \vdash J_1 \qquad \cdots \qquad \Gamma_n \vdash J_n}{\Gamma \vdash J}$$

Identity types

Formation rule

$$\frac{A \in \text{Type} \qquad a \in A \qquad b \in A}{\text{Id}_A(a, b) \in \text{Type}}$$

For example, if $a \in A$ then $\text{Id}_A(a, a) \in \text{Type}$

Introduction rule

$$\frac{a \in A}{\text{r}(a) \in \text{Id}_A(a, a)}$$

Elimination rule

$$\frac{\begin{array}{l} p \in \mathrm{Id}_A(a,b) \\ x \in A\,, y \in A\,, u \in \mathrm{Id}_A(x,y) \vdash C(x,y,u) \in \mathrm{Type} \\ x \in A \vdash c(x) \in C(x,x,\mathrm{r}(x)) \end{array}}{\mathrm{J}(a,b,p,c) \in C(a,b,p)}$$

Idea:

$$\frac{a = b \qquad \begin{array}{c} [x \in A] \\ \vdots \\ C(x,x) \end{array}}{C(a,b)}$$

Cf. Lawvere's treatment of equality in categorical logic.

Computation rule

$$\frac{\begin{array}{l} a \in A \\ x \in A\,, y \in A\,, u \in \mathrm{Id}_A(x,y) \vdash C(x,y,u) \in \mathrm{Type} \\ x \in A \vdash c(x) \in C(x,x,\mathrm{r}(x)) \end{array}}{\mathrm{J}(a,a,\mathrm{r}(a),c) = c(a) \in C(a,a,\mathrm{r}(a))}$$

Idea:

$$\dfrac{\dfrac{a \in A}{a = a} \qquad \begin{matrix}[x \in A] \\ \vdots \\ C(x,x)\end{matrix}}{C(a,a)} \qquad \longrightarrow \qquad \begin{matrix} a \in A \\ \vdots \\ \vdots \\ C(a,a)\end{matrix}$$

Definitional equality vs. propositional equality

Definition. We say that $a, b \in A$ are propositionally equal if there exists $p \in \mathrm{Id}_A(a, b)$.

Theorem (Hofmann and Streicher)**.**

$$a = b \in A \quad \overset{\Rightarrow}{\not\Leftarrow} \quad \text{There exists } p \in \mathrm{Id}_A(a, b)$$

Proposition (Hofmann)**.** Adding the rules

$$\frac{p \in \mathrm{Id}_A(a, b)}{a = b \in A} \qquad \frac{p \in \mathrm{Id}_A(a, b)}{p = \mathrm{r}(a) \in \mathrm{Id}_A(a, b)}$$

makes type-checking undecidable.

Weakness of propositional equality

We have

$$\frac{p \in \mathrm{Id}_A(a,b) \qquad q \in \mathrm{Id}_A(b,c)}{q \circ p \in \mathrm{Id}_A(a,c)}$$

The rule

$$\frac{p \in \mathrm{Id}_A(a,b) \qquad q \in \mathrm{Id}_A(b,c) \qquad r \in \mathrm{Id}_A(c,d)}{(r \circ q) \circ p = r \circ (q \circ p) \in \mathrm{Id}_A(a,d)}$$

does not seem derivable, but only

$$\frac{p \in \mathrm{Id}_A(a,b) \qquad q \in \mathrm{Id}_A(b,c) \qquad r \in \mathrm{Id}_A(c,d)}{\alpha \in \mathrm{Id}_{\mathrm{Id}_A(a,d)}\big((r \circ q) \circ p\,, r \circ (q \circ p)\big)}$$

Part II

The identity type weak factorisation system

Types as spaces

Idea

- Elements $a \in A$ as points
- Elements $p \in \mathrm{Id}_A(a, b)$ as paths from a to b
- Elements $\alpha \in \mathrm{Id}_{\mathrm{Id}_A(a,b)}(p, q)$ as homotopies from p to q
- ...

Examples

$$\frac{a \in A}{\mathrm{r}(a) \in \mathrm{Id}_A(a, a)} \qquad \frac{p \in \mathrm{Id}_A(a, b) \qquad q \in \mathrm{Id}_A(b, c) \qquad r \in \mathrm{Id}_A(c, d)}{\alpha \in \mathrm{Id}_{\mathrm{Id}_A(a,d)}\big((r \circ q) \circ p\,, r \circ (q \circ p)\big)}$$

The syntactic category **ML**

- **Objects.** Types $A, B, C, \ldots$.
- **Maps.** Terms-in-context, i.e. $f : X \to A$ is

$$x \in X \vdash f(x) \in A\,.$$

Examples. For $A \in \mathrm{Type}$, let

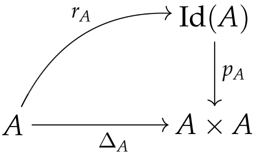
$$\mathrm{Id}(A) = \textstyle\sum_{x,y \in A} \mathrm{Id}_A(x,y)$$

We have maps

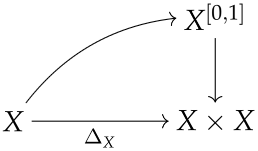
$$\begin{array}{ccc} A & \xrightarrow{r_A} & \mathrm{Id}(A)\,, \\ x & \longmapsto & (x, x, \mathrm{r}(x)) \end{array} \qquad \begin{array}{ccc} \mathrm{Id}(A) & \xrightarrow{p_A} & A \times A \\ (x, y, u) & \longmapsto & (x, y) \end{array}$$

Identity types as path spaces

- **ML**

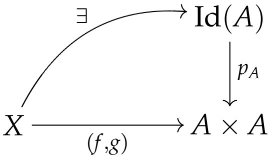


- **Top**

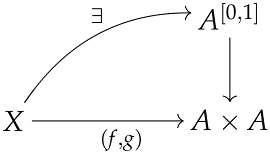


Propositional equality as homotopy

- Terms $x \in X \vdash f(x)\,, g(x) \in A$ are propositionally equal iff



- Maps $f : X \to A$ and $g : X \to A$ in **Top** are homotopic iff

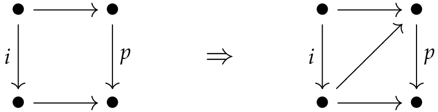


Still missing: elimination and computation rules.

Lifting properties

Let $\mathbb{C}$ be a category.

Definition. Let i and p be maps in $\mathbb{C}$. We say that i has the left lifting property with respect to p if



Notation: $i \pitchfork p$.

Let S be a class of maps in $\mathbb{C}$. Define

$$^{\pitchfork}S = \{\, i \mid (\forall p \in S)\ i \pitchfork p \} \qquad S^{\pitchfork} = \{\, p \mid (\forall i \in S)\ i \pitchfork p \} \,.$$

Example: fibrations

Definition. A continuous map $p : B \to A$ is a fibration if it has the homotopy lifting property, i.e. every diagram

$$\begin{array}{ccc} X \times \{0\} & \longrightarrow & B \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle p} \\ X \times [0,1] & \longrightarrow & A \end{array}$$

has a diagonal filler.

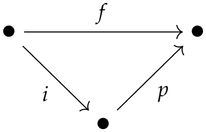
$$\{\, \text{Fibrations} \,\} = \{\, i_X \mid X \in \mathbf{Top} \,\}^{\pitchfork}$$

Example. $p : A^{[0,1]} \to A \times A.$

Weak factorization systems

Definition (Bousfield 1977)**.** A weak factorization system on $\mathbb{C}$ is a pair (L, R) of classes of maps such that:

(1) Every map f in $\mathbb{C}$ factors as



with $i \in L\,,\ p \in R\,.$

(2) $L = {}^{\pitchfork}R\,,\ R = L^{\pitchfork}\,.$

Example. The category **Top** has a w.f.s. (L, R) where

$$R = \{\text{Fibrations}\}\,, \quad L \subseteq \{\text{Homotopy equivalences}\}\,.$$

Examples. Quillen model structures.

Projections in **ML**

Definition. A map in **ML** is a projection if it has the form

$$\begin{array}{rccc} p: & \sum_{x \in A} B(x) & \to & A \\ & (x, y) & \mapsto & x \end{array}$$

where $x \in A \vdash B(x) \in \text{Type}$.

Example. Recall

$$\text{Id}(A) = \sum_{x, y \in A} \text{Id}_A(x, y)\,.$$

We have the projection

$$\begin{array}{rccc} p_A: & \text{Id}(A) & \longrightarrow & A \times A \\ & (x, y, u) & \longmapsto & (x, y) \end{array}$$

The identity type weak factorisation system

Theorem (Gambino and Garner)**.** The syntactic category **ML** has a weak factorisation system (L, R) given by

$$L = {}^{\pitchfork}P\,, \quad R = L^{\pitchfork}\,.$$

where $P = \{$ Projections $\}$.

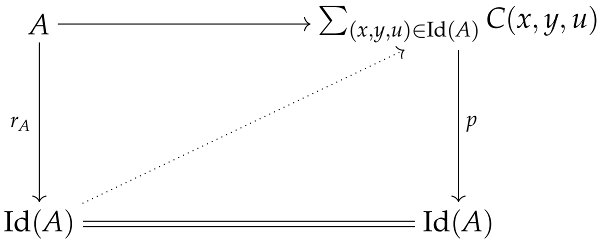
Note. L-maps and R-maps can be characterized explicitly.

Example. The diagonal $\Delta_A : A \to A \times A$ factors as

$$A \xrightarrow{r_A} \mathrm{Id}(A) \xrightarrow{p_A} A \times A$$

To show: $\{r_A\} \pitchfork P$.

It suffices to consider



Top horizontal arrow gives

$$x \in A \vdash c(x) \in C(x, x, \mathrm{r}(x))$$

So, we can apply the elimination rule:

$$\frac{x \in A \vdash c(x) \in C(x, x, \mathrm{r}(x))}{x \in A, y \in A, u \in \mathrm{Id}_A(x, y) \vdash \mathrm{J}(x, y, u, c) \in C(x, y, u)}$$

Top triangle commutes by computation rule.

Homotopy-theoretic models

Theorem (Awodey and Warren)**.** The rules for identity types admit an interpretation in every category $\mathbb{C}$ with a w.f.s. (L, R).

Idea.

- Dependent types as R-maps

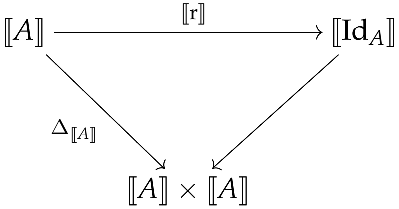
$$x \in A \vdash B(x) \in \mathsf{Type} \qquad \Longrightarrow \qquad \begin{array}{c} [\![B]\!] \\ \big\downarrow \\ [\![A]\!] \end{array}$$

- Terms as sections

$$x \in A \vdash b(x) \in B(x) \qquad \Longrightarrow \qquad \begin{array}{ccc} [\![A]\!] & \xrightarrow{[\![b]\!]} & [\![B]\!] \\ & {}_{1}\searrow \quad \swarrow & \\ & [\![A]\!] & \end{array}$$

- Identity types as path objects

$$\left.\begin{array}{c} x \in A\,, y \in A \vdash \mathrm{Id}_A(x,y) \in \mathrm{Type} \\[1em] x \in A \vdash \mathrm{r}(x) \in \mathrm{Id}_A(x,x) \end{array}\right\} \Longrightarrow$$



Elimination terms given by diagonal fillers.

Note. Coherence issues (Warren, van den Berg and Garner).

Part III

The fundamental weak ω-groupoid of a type

The fundamental groupoid $\pi_1(A)$ of a type A

- **Objects.** Elements $a, b, \ldots \in A$
- **Maps.** Equivalence classes $[p] : a \to b$, where $p \in \mathrm{Id}_A(a, b)$ and

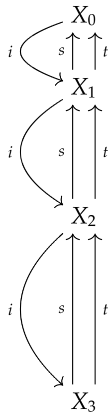
$$p \sim q \Leftrightarrow \text{ there exists } \alpha \in \mathrm{Id}_{\mathrm{Id}_A(a,b)}(p, q) \,.$$

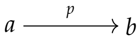
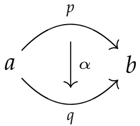
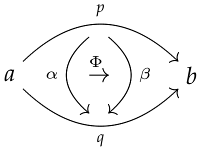
$\approx$ Fundamental groupoid of a space.

Question

- What happens if we do not quotient identity proofs?

The globular set $\pi(A)$ of a type A



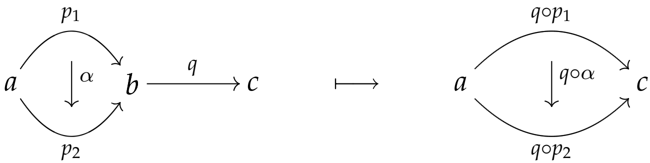
X	$\pi(A)$	
X_0	A	$a, b, \dots$
X_1	$\mathrm{Id}_A(a,b)$	
X_2	$\mathrm{Id}_{\mathrm{Id}_A(a,b)}(p,q)$	
X_3	$\mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_A(a,b)}(p,q)}(\alpha,\beta)$	
$\dots$	$\dots$	$\dots$

Weak ω-groupoids

Definition (Batanin 1998, Leinster 2004).

Weak ω-category = Globular set + action by a contractible operad
= Globular set + 'composition operations'

Example.



We have a weak ω-groupoid if all n-cells have weak inverses.

The weak ω-groupoid of a type

Theorem (Garner and van den Berg, Lumsdaine)**.**

For every type A, the globular set $\pi(A)$ is a weak ω-groupoid.

Examples.

$$\frac{\alpha \in \mathrm{Id}_{\mathrm{Id}_A(a,b)}(p_1, p_2) \qquad q \in \mathrm{Id}_A(b,c)}{q \circ \alpha \in \mathrm{Id}_{\mathrm{Id}_A(a,c)}(q \circ p_1, q \circ p_2)}$$

$$\frac{p \in \mathrm{Id}_A(a,b)}{p^{-1} \in \mathrm{Id}_A(b,a)} \qquad \frac{p \in \mathrm{Id}_A(a,b)}{\theta_p \in \mathrm{Id}_{\mathrm{Id}_A(a,a)}(p^{-1} \circ p, \mathrm{r}(a))}$$

Open problems

Models

- Models in weak ω-groupoids

Relationship with homotopy theory

- Simplicial identity types

Relationship with higher categories

- Free higher categories from syntax