Exercise Sheet 1

AILA Summer School

August 23rd, 2023

1. Prove in detail that the following are indeed categories:
   (a) **Grp**, the category of groups and group homomorphisms;
   (b) **Top**, the category of topological spaces and continuous functions;
   (c) Another example of your choice.
2. Find a terminal object in the following categories:
   (a) **Set**, the category of sets and functions;
   (b) **Rel**, the category of sets and relations.
3. Let $\mathbb{C}$ be a category, $f\colon X \to Y$ a map in $\mathbb{C}$. An **inverse** of f is a map $g\colon Y \to X$ such that

   $$g \circ f = 1_X\,, \quad f \circ g = 1_Y\,.$$

   We say that f is an **isomorphism** if it admits an inverse.
   (a) Prove that an inverse of f, if it exists, is unique.
   (b) Prove that the composite of two isomorphisms is an isomorphism and that identities are isomorphisms.
   (c) Prove that functors preserve isomorphisms, i.e. that if $f\colon X \to Y$ is an isomorphism in $\mathbb{C}$ and $F\colon \mathbb{C} \to \mathbb{D}$ is a functor, then $Ff\colon FX \to FY$ is an isomorphism in $\mathbb{D}$.
   (d) Let $\mathbb{C}$ be a category and $f\colon X \to Y$ an isomorphism in $\mathbb{C}$. Prove that for every $Z \in \mathbb{C}$, the functions defined by composition with f

   $$\mathbb{C}[Y,Z] \xrightarrow{(-)\circ f} \mathbb{C}[X,Z] \qquad \mathbb{C}[Z,X] \xrightarrow{f\circ(-)} \mathbb{C}[Z,Y]$$

   are bijections.
4. Prove that a terminal object in a category, if it exists, is unique up to unique isomorphism.
5. (a) Let $F\colon \mathbb{C} \to \mathbb{D}$, $F'\colon \mathbb{C} \to \mathbb{D}$ be functors and $\alpha\colon F \Rightarrow F'$ be a natural transformation between them. For a functor $G\colon \mathbb{D} \to \mathbb{E}$, define a family of maps $GFX \to GF'X$ in $\mathbb{E}$, for $X \in \mathbb{C}$, and prove that this forms a natural transformation, which we will denote $G\alpha\colon GF \Rightarrow GF'$.
   (b) Let $G\colon \mathbb{D} \to \mathbb{E}$, $G'\colon \mathbb{D} \to \mathbb{E}$ be functors and $\beta\colon G \Rightarrow G'$ be a natural transformation between them. For a functor $F\colon \mathbb{C} \to \mathbb{D}$, define a family of maps $GFX \to G'FX$ in $\mathbb{E}$, for $X \in \mathbb{C}$, and prove that it is a natural transformation, which we will denote $\beta F\colon GF \Rightarrow G'F$.

(c) Let $F\colon \mathbb{C} \to \mathbb{D}$, $F'\colon \mathbb{C} \to \mathbb{D}$, $G\colon \mathbb{D} \to \mathbb{E}$, $G'\colon \mathbb{D} \to \mathbb{E}$ be functors, $\alpha\colon F \Rightarrow F'$ and $\beta\colon G \Rightarrow G'$ be natural transformations. By part (a), we have natural transformations

$$G\alpha\colon GF \Rightarrow GF'\,, \qquad G'\alpha\colon G'F \Rightarrow G'F'\,.$$

By part (b), we also have natural transformations

$$\beta F\colon GF \Rightarrow G'F\,, \qquad \beta F'\colon GF' \Rightarrow G'F'\,.$$

Prove that the following diagram of natural transformations commutes:

$$\begin{CD} GF @>{G\alpha}>> GF' \\ @V{\beta F}VV @VV{\beta F'}V \\ G'F @>>{G'\alpha}> G'F'\,. \end{CD}$$

We will denote the value of the composite as $\beta\alpha\colon GF \Rightarrow G'F'$.