Lezione 6 : alcune variazioni / estensioni e temi di ricerca.

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λ-calcolo semplicemente tipato — Categorie cartesiane chiuse

teorie dipendenti dei tipi — categorie localmente cartesiane chiuse

$$(\Pi x : A)\, B(x)$$

$$(\Sigma x : A)\, B(x)$$

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}
\qquad
\begin{tikzcd} \mathbb{C}/A \arrow[d, bend right=60, "\Sigma_f"'] \arrow[d, bend left=60, "\Pi_f"] \\ \mathbb{C}/B \arrow[u, "f^*"'] \end{tikzcd}$$

⚠ substitution.

B. Jacobs.

teorie dipendenti dei tipi con identity types

categorie loc. cartesiane chiuse + ...

$$\frac{A : \text{type} \qquad a : A \qquad b : A}{\mathrm{Id}_A(a,b) : \text{type}}$$

$$\begin{tikzcd} \mathrm{Id}_A \arrow[d] \\ A \times A \end{tikzcd}$$

Idea : $p : \mathrm{Id}_A(a,b) \rightsquigarrow$

"p prova che $a = b$"

$$\frac{a : A}{\mathrm{refl}(a) : \mathrm{Id}_A(a,a)}$$

$$\begin{tikzcd} A \arrow[r, "\mathrm{refl}_A"] \arrow[dr, bend right, "\Delta_A"'] & \mathrm{Id}_A \arrow[d] \\ & A \times A \end{tikzcd}$$

$a \mapsto (a,a)$

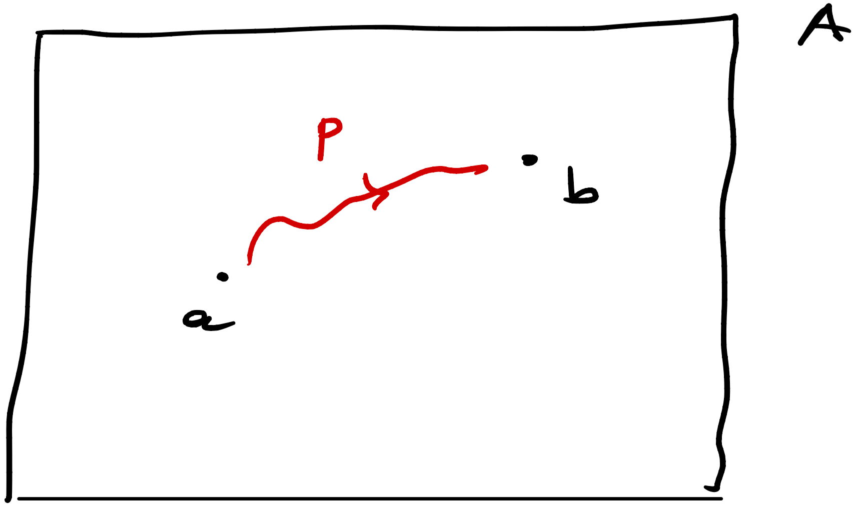
$\mathrm{refl}(a) :$

$\mathrm{Id}(a,a)$

$$\frac{p : \mathrm{Id}(a,b) \qquad q : \mathrm{Id}(a,b)}{\mathrm{Id}_{\mathrm{Id}(a,b)}(p,q)}$$

categorie lcc
+
nozione di omotopia
(Homotopy Type Theory)

$p : \mathrm{Id}_A(a,b)$



regole per tipi identità

sistema di fattorizzazione debole

Cf anche Voevodsky "Univalent Foundations"

Assioma di Univalenza

- Dati tipi A, B, possiamo introdurre una nozione di equivalenza / isomorfismo tra A e B

$$
\begin{tikzcd}
A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"]
\end{tikzcd}
\qquad
\begin{aligned}
\eta &: (\Pi x : A)\, \mathrm{Id}_A(x, gfx) \quad \ldots \\
\varepsilon &: (\Pi y : B)\, \mathrm{Id}_B(y, fgy)
\end{aligned}
$$

$\Rightarrow$ Possiamo formare un tipo $\mathrm{Equiv}(A, B)$ i cui elementi sono equivalenze tra A e B.

- Se abbiamo un tipo universo $\mathcal{U}$, con $A, B : \mathcal{U}$

$$\frac{\mathcal{U} : \text{type} \qquad A : \mathcal{U} \qquad B : \mathcal{U}}{\mathrm{Id}_{\mathcal{U}}(A, B)}$$

$p : \mathrm{Id}_{\mathcal{U}}(A, B) \iff p$ prova che A e' uguale a B.

- $$\mathrm{Id}_{\mathcal{U}}(A, B) \xrightarrow{\Phi} \mathrm{Equiv}(A, B)$$

e.g. $$\mathrm{Id}_{\mathcal{U}}(A, A) \longrightarrow \mathrm{Equiv}(A, A)$$

$$\mathrm{refl}(A) \longmapsto 1_A : A \to A$$

Assioma di Univalenza : Φ e' un' equivalenza.

2-categorie e ∞-categorie

Def. Una 2-categoria $\mathbb{C}$

- oggetti $Ob(\mathbb{C})$
- $\forall A, B \in Ob(\mathbb{C})$, una categoria $\mathbb{C}(A,B)$

$\vdots$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

$f, g \in Ob(\mathbb{C}(A,B))$

$\alpha : f \Rightarrow g$ in $\mathbb{C}(A,B)$

$$\mathrm{app}\big((\lambda x : A)\, b,\ a\big) \Longrightarrow b[a/x]$$

$$\begin{tikzcd}
X \times A \arrow[d, "\lambda(b) \times 1"'] \arrow[drr, bend left, "b"] \arrow[dr, phantom, "\Uparrow"] & & \\
B^A \times A \arrow[rr, "\mathrm{app}"'] & & B
\end{tikzcd}$$

"Two-dimensional categorical logic".

S. Lack, A 2-categories companion, ArXiv.

J. Lurie, Higher topos theory

↳ Categorical Logic.