Differential λ-calculus and analytic functors

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Overview

Goal. Define a model of the

- differential λ-calculus [Ehrhard and Regnier]

using

- anaytic functors [Joyal]

Based on joint works with M. Fiore, M. Hyland, A. Joyal, G. Winskel.

Key idea: $$f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \quad \Rightarrow \quad f'(x) = \sum_{n=0}^{\infty} f_{n+1} \frac{x^n}{n!}$$

Outline of the talk

Part I: Syntax

- Review of λ-calculus
- Differential λ-calculus

Part II: The relational model

- Relations
- Differential structure

Part III: Analytic functors

- Profunctors
- Differential structure

Part I: Syntax

Simply-typed λ-calculus

Product types

$$\frac{a : A \qquad b : B}{\mathrm{pair}(a, b) : A \times B} \qquad\qquad \frac{c : A \times B}{\pi_1(c) : A} \qquad \frac{c : A \times B}{\pi_2(c) : B}$$

Function types

$$\frac{x : A \vdash b : B}{(\lambda x : A) b : B^A} \qquad \frac{f : B^A \quad a : A}{\mathrm{app}(f, a) : B}$$

The β-rule

$$\mathrm{app}((\lambda x : A) b, a) = b[a/x] : B$$

Cartesian closed categories

$\mathcal{C}$ a cartesian closed category.

Binary products

- There are $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that

$$\mathcal{C}[X, A \times B] \xrightarrow{(\pi_1,\pi_2)\circ(-)} \mathcal{C}[X, A] \times \mathcal{C}[X, B]$$

  is a bijection.

Exponentials

- There is $\mathrm{app} : B^A \times A \to B$ such that

$$\mathcal{C}[X, B^A] \xrightarrow{(-)\times A} \mathcal{C}[X \times A, B^A \times A] \xrightarrow{\mathrm{app}\circ(-)} \mathcal{C}[X \times A, B]$$

  is a bijection.

Substitution

Syntax. For $x : A \vdash b : B$ and $\Gamma \vdash a : A$,

- $b[a/x]$ is defined by structural induction on b.

Semantics. For $b : A \to B$ and $a : \Gamma \to A$,

- $b[a/x]$ is given by composition

$$\Gamma \xrightarrow{a} A \xrightarrow{b} B$$

The naturality of the adjunctions ensures that these match.

The differential λ-calculus (I)

For a differentiable map $f : A \to B$ and $x \in A$, we have a linear map (the Jacobian):

$$\begin{array}{rccc} f'(x): & A & \longrightarrow & B \\ & a & \longmapsto & f'(x) \cdot a \end{array} \tag{1}$$

Transposing, for $a \in A$, we obtain a (generally) non-linear map:

$$\begin{array}{rccc} f'(x): & A & \longrightarrow & B \\ & x & \longmapsto & f'(x) \cdot a \end{array} \tag{2}$$

Differentiation rule [Ehrhard-Regnier]**.** This corresponds to (2).

$$\frac{\Gamma \vdash f : B^A \qquad \Delta \vdash a : A}{\Gamma, \Delta \vdash \mathrm{D} f \cdot a : B^A}$$

We think of a as being applied **linearly**.

The differential λ-calculus (II)

Fix a commutative rig R and allow linear combinations of λ-terms.

The differential β-rule:

$$\mathrm{D}\big((\lambda x : A)b\big) \cdot a = \lambda x\left(\frac{\partial b}{\partial x} \cdot a\right),$$

where $\dfrac{\partial b}{\partial x} \cdot a$ is the linear substitution of a in b.

This is defined by structural induction on b.

Idea: "Sum of all terms obtained by replacing one linear occurrence of x in b with a."

Models

Questions

- How to find models?
- What equations are needed in order to model linear substitution?

Concrete models

- Köthe spaces (some topological vector spaces) [Ehrhard]
- Finiteness spaces [Ehrhard]
- Relational model [Blute, Cockett, Seely], [Ehrhard], [Hyland]

Categorical axiomatisations. Differential categories and variants:

- [Blute, Cockett, Seely]
- [Fiore]
- [Lemay], [Cockett and Lemay]
- [Manzonetto]

Part II: The relational model

The category of relations

Define the category **Rel** as follows.

- **Objects:** sets
- **Morphisms:** relations

$$F : A \to B \quad \text{is} \quad F \subseteq B \times A$$

- **Composition:** for $A \xrightarrow{F} B \xrightarrow{G} C$ we define

$$(G \circ F)(c, a) = (\exists b \in B)\ G(c, b) \wedge F(b, a)$$

- **Identity:** for a set A, we define $1_A : A \to A$ by

$$1_A(a, b) = \begin{cases} 1 & \text{if } a = b\,, \\ 0 & \text{otherwise.} \end{cases}$$

Structure of **Rel**

- Symmetric monoidal structure: $A \times B$
- Closed structure (internal hom): $A \multimap B =_{\mathrm{def}} B \times A$, since

$$\mathbf{Rel}[A \times B, C] \cong \mathbf{Rel}[A, B \multimap C]$$

- Binary products: $A + B$, since

$$\mathbf{Rel}[X, A] \times \mathbf{Rel}[X, B] \cong \mathbf{Rel}[X, A + B]$$

- Terminal object: 0, since

$$\mathbf{Rel}[X, 0] \cong 1\,.$$

The exponential modality

For $A \in \mathbf{Rel}$, define

$$\begin{aligned} !A &= \text{free commutative monoid on } A \\ &= \text{set of multisets } \alpha = [a_1, \ldots, a_n] \text{ of elements of } A \end{aligned}$$

This is a comonad, with

$$\mathsf{d}_A : !A \to A \qquad \mathsf{p}_A : !A \to !!A$$

defined by

$$\mathsf{d}_A(a, \alpha) \Leftrightarrow [a] = \alpha \qquad \mathsf{p}_A([\alpha_1, \ldots, \alpha_n], \alpha) \Leftrightarrow \alpha_1 + \ldots + \alpha_n = \alpha$$

Seely isomorphisms:

- $!(A + B) \cong !A \times !B$ and $!0 = 1$.

Theorem [Folklore]**.** **Rel** is a (degenerate) model of classical linear logic.

The Kleisli category

Define the category $\mathbf{Rel}_!$ as follows.

- **Objects:** sets
- **Morphisms:** relations $F : !A \to B$
- **Composition:** given $F : !A \to B$ and $G : !B \to C$, consider

$$!A \xrightarrow{\mathsf{p}_A} !!A \xrightarrow{!F} !B \xrightarrow{G} C$$

- **Identity:** for A, consider $\mathsf{d}_A : !A \to A$.

Idea:

- $\mathbf{Rel}$ = sets and linear maps, $\mathbf{Rel}_!$ = sets and non-linear maps
- Given $F : A \to B$ linear, we obtain $!A \xrightarrow{\mathsf{d}_A} A \xrightarrow{F} B$
- $\mathbf{Pt}(A) =_{\mathrm{def}} \mathbf{Rel}_![0, A] = \mathbf{Rel}[!0, A] = \mathbf{Rel}[1, A] = \mathbf{Pow}(A)$.

The cartesian closed structure

- Binary products

$$\begin{aligned}\mathbf{Rel}_!\,[X,A]\times\mathbf{Rel}_!\,[X,B] &= \mathbf{Rel}\,[!X,A]\times\mathbf{Rel}\,[!X,B]\\ &\cong \mathbf{Rel}[!X,A+B]\end{aligned}$$

- Exponentials

$$\begin{aligned}\mathbf{Rel}_!\,[X+A,B] &= \mathbf{Rel}\,[!(X+A),B]\\ &\cong \mathbf{Rel}\,[!X\times !A,B]\\ &\cong \mathbf{Rel}\,[!X,!A\multimap B]\\ &\cong \mathbf{Rel}\,[!X,!A\multimap B]\\ &= \mathbf{Rel}_!\,[X,B^A]\end{aligned}$$

where $B^A =_{\mathrm{def}} !A \multimap B$.

Differential structure (I)

Want:

$$\frac{F : !A \to B}{\mathrm{d}F : !A \times A \to B}$$

Idea (Differential categories). [Blute, Cockett, Seely]

- it suffices to have maps $\partial_A : !A \times A \to !A$. Then $\mathrm{d}F$ is obtained as

$$!A \times A \xrightarrow{\partial_A} !A \xrightarrow{F} B$$

- it suffices to have maps $\bar{d}_A : A \to !A$. Then $\mathrm{d}F$ is obtained as

$$!A \times A \xrightarrow{1 \times \bar{d}_A} !A \times !A \xrightarrow{\bar{c}_A} !A \xrightarrow{F} B$$

Differential structure (II)

Explicitly:

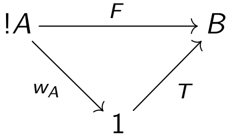
- define $\bar{d}_A : A \to !A$ by
$$\bar{d}_A(\alpha, a) \Leftrightarrow \alpha = [a]$$
- define $\partial_A : !A \times A \to !A$ by
$$\partial_A(\beta, (\alpha, a)) \Leftrightarrow \beta = \alpha + [a]$$
- define $\mathrm{d}F : !A \times A \to B$ by
$$\mathrm{d}F(b, (\alpha, a)) \Leftrightarrow F(b, \alpha + [a])\,.$$

Note: Shift of one from α to $\alpha + [a]$.

Theorem. [BCS], [Ehrhard], [Hyland] The rules of differential λ-calculus hold in $\mathbf{Rel}_!$.

Differential structure (III)

Example. Say $F : !A \to B$ is **constant** if there is $T \subseteq B$ such that



in **Rel**. This means

$$F(b, \alpha) \quad \Leftrightarrow \quad w_A(*, \alpha) \wedge T(b, *) \quad \Leftrightarrow \quad \alpha = [\,] \wedge b \in T$$

Proposition. If F constant, then $\mathrm{d}F = 0$.

Proof. $\mathrm{d}F(b, (\alpha, a)) \quad \Leftrightarrow \quad F(b, \alpha + [a]) \quad \Leftrightarrow \quad \alpha + [a] = [\,] \wedge b \in T \quad \Leftrightarrow \quad \bot$

Note. The axioms can be expressed in terms of d_A.

Part III: Analytic functors

Profunctors

A ‘categorification’ of relations [Bénabou], [Lawvere].

Definition. Let A, B be small categories. A (B, A)-**profunctor** is a functor

$$F : B^{\mathrm{op}} \times A \to \mathbf{Set}$$

Some intuition

- $F(b, a)$ is the set of ‘proofs’ that b and a are related.
- Sets $F(b, a)$, together with actions

$$F(b, a) \times A[a, a'] \to F(b, a')\,, \quad B[b', b] \times F(b, a) \to F(b', a)$$

Example. For a small category A, we have

$$A[-, -] : A^{\mathrm{op}} \times A \to \mathbf{Set}\,.$$

The bicategory of profunctors

Define the bicategory **Prof** as follows.

- **Objects:** small categories
- **Morphisms:** profunctors

$$F : A \to B \quad \text{is} \quad F : B^{\mathrm{op}} \times A \to \mathbf{Set}$$

- **2-cells:** natural transformations
- **Horizontal composition:** for $A \xrightarrow{F} B \xrightarrow{G} \mathbb{C}$ we define

$$(G \circ F)(c, a) = \int^{b \in B} G(c, b) \times F(b, a)$$

- **Identity:** for a small category A, we define $1_A : A \to A$ by

$$1_A(b, a) = A[b, a]$$

The structure of **Prof**

- Symmetric monoidal structure: $A \times B$
- Closed structure (internal hom): $A \multimap B =_{\mathrm{def}} B \times A^{\mathrm{op}}$, since

$$\mathbf{Prof}[X \times A, B] \cong \mathbf{Prof}[X, A \multimap B]$$

- Binary products: $A + B$, since

$$\mathbf{Prof}[X, A] \times \mathbf{Prof}[X, B] \cong \mathbf{Prof}[X, A + B]$$

- Terminal object: 0, since

$$\mathbf{Prof}[X, 0] \cong 1$$

Note: All this is now in a 'weak', bicategorical, sense.

The exponential pseudo-comonad

For $A \in \mathbf{Prof}$, define $!A =$ free symmetric monoidal category on A as follows.

- **Objects:** $(a_1, \dots, a_n)$, where $n \in \mathbb{N}$ and $a_i \in A$,
- **Maps:** $(\sigma, f_1, \dots f_n) : (a_1, \dots, a_n) \to (b_1, \dots, b_m)$, only if $n = m$, with $\sigma \in \mathrm{S}_n$ and $f_i : a_i \to b_{\sigma(i)}$.

This is a pseudocomonad, with

$$\mathrm{d}_A : !A \to A \qquad \mathrm{p}_A : !A \to !!A$$

defined by

$$\mathrm{d}_A(a, \alpha) = !A[\alpha, (a)] \qquad \mathrm{p}_A((\alpha_1, \dots, \alpha_n), \alpha) = !A[\alpha, \alpha_1 \otimes \dots \otimes \alpha_n]$$

Seely equivalences:

- $!(A + B) \simeq !A \times !B$ (equivalences, not isos) and $!0 \cong 1$

The Kleisli bicategory

Define the bicategory $\mathbf{Prof}_!$ as follows.

- **Objects:** small categories
- **Morphisms:** profunctors $F : !A \to B$
- **2-cells:** natural transformations
- **Horizontal composition:** for $F : !A \to B$ and $G : !B \to C$, consider

$$!A \xrightarrow{\mathsf{p}_A} !!A \xrightarrow{!F} !B \xrightarrow{G} C$$

- **Identity:** for A, consider $\mathsf{d}_A : !A \to A$.

Idea:

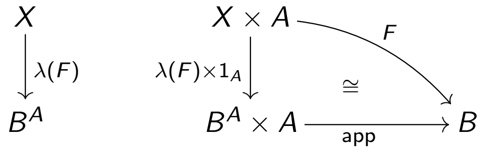
- $\mathbf{Prof}$ = categories and linear maps
- $\mathbf{Prof}_!$ = categories and non-linear maps
- $\mathbf{Pt}(A) =_{\mathrm{def}} \mathbf{Prof}_![0, A] = \mathbf{Prof}[!0, A] = \mathbf{Prof}[1, A] = [A^{\mathrm{op}}, \mathbf{Set}]$.

The cartesian closed structure

In analogy with the relational model, we obtain the following.

Theorem [FGHW] The bicategory $\mathbf{Prof}_!$ is cartesian closed.

This means that, for $F : X \times A \to B$, there are 2-cells



witnessing the β-rule of the λ-calculus:

$$\mathrm{app}\big((\lambda x : A)F, x\big) \cong F$$

Symmetric sequences (I)

The maps in $\mathbf{Prof}_!$ are of independent interest.

Consider $A = B = 1$.

$$
\begin{aligned}
F : 1 \to 1 \text{ in } \mathbf{Prof}_! \quad &= \quad F : !1 \to 1 \quad \text{profunctor} \\
&= \quad F : 1^{\mathrm{op}} \times !1 \to \mathbf{Set} \quad \text{functor} \\
&= \quad F : \mathbf{P} \to \mathbf{Set} \quad \text{functor}
\end{aligned}
$$

where $\mathbf{P}$ is the category of natural numbers and permutations.

Explicitly, for every $n \in \mathbb{N}$, a set $F(n)$ with a S_n-action.

These functors are called **symmetric sequences**.

Analytic functors (I)

Let $F : \mathbf{P} \to \mathbf{Set}$ be a symmetric sequence.

The **analytic functor** associated to F is the functor $\widehat{F} : \mathbf{Set} \to \mathbf{Set}$ defined by

$$\widehat{F}(X) = \sum_{n \in \mathbb{N}} \frac{F(n) \times X^n}{S_n}$$

Compare with

$$f(x) = \sum_{n \in \mathbb{N}} f_n \frac{x^n}{n!}$$

Analytic functors support a rich calculus, including differentiation. [Joyal].

Symmetric sequences (II)

Let A and B be small categories.

$$\begin{aligned} F : A \to B \text{ in } \mathbf{Prof}_! \quad &= \quad F : !A \to B \text{ profunctor} \\ &= \quad F : B^{\mathrm{op}} \times !A \to \mathbf{Set} \text{ functor} \end{aligned}$$

Such an F

$$(b, a_1, \ldots, a_n) \mapsto F(b, a_1, \ldots, a_n)$$

We call these (B, A)-**symmetric sequences**.

A 'many-sorted' version of symmetric sequences.

Analytic functors (II)

Let F be a (B, A)-symmetric sequence. The **analytic functor** $\widehat{F}$ is defined by

$$\begin{array}{ccc}
\mathbf{Prof}_![0, A] & \xrightarrow{F \circ (-)} & \mathbf{Prof}_![0, B] \\
\| & & \| \\
\mathbf{Pt}(A) & & \mathbf{Pt}(B) \\
\| & & \| \\
[A^{\mathrm{op}}, \mathbf{Set}] & \xrightarrow[\widehat{F}]{} & [B^{\mathrm{op}}, \mathbf{Set}]
\end{array}$$

Explicitly:

$$\widehat{F}(X)(b) = \int^{(a_1,\ldots,a_n) \in !A} F(b, a_1, \ldots, a_n) \times X(a_1) \times \ldots \times X(a_n)$$

Theorem [FGHW], [Fiore] Analytic functors support a rich calculus, including differentiation.

Differential structure

Differentiation. Define

$$\bar{d}_A : A \to !A$$

by

$$\bar{d}_A(\alpha, a) = !A[\alpha, (a)]$$

For $F : !A \to B$, define

$$\mathrm{d}F : !A \times A \to B$$

by

$$!A \times A \xrightarrow{1 \times \bar{d}_A} !A \times !A \xrightarrow{\bar{c}_A} !A \xrightarrow{F} B$$

so that

$$\mathrm{d}F(b, (\alpha, a)) = F(b, \alpha \otimes [a])\,.$$

Theorem. All the rules of differential λ-calculus are valid in $\mathbf{Prof}_!$, up to isomorphism.

Differentiation of analytic functors

$$\dfrac{\dfrac{F : !A \times A \to B}{\mathrm{d}F : !A \times A \to B}}{\mathrm{d}F : !A \to (A \multimap B)}$$

Recalling $A \multimap B = B \times A^{\mathrm{op}}$, we obtain

$$\widehat{\mathrm{d}F} : [A^{\mathrm{op}}, \mathbf{Set}] \to [B^{\mathrm{op}} \times A, \mathbf{Set}]$$

Explicitly

$$\widehat{\mathrm{d}F}(X)(b, a) = \int^{(a_1,\ldots,a_n)\in !A} F(b, a_1, \ldots, a_n, a) \times X(a_1) \times \ldots \times X(a_n)$$

Note. Shift of one from $(a_1, \ldots, a_n)$ to $(a_1, \ldots, a_n, a)$.

Calculus of analytic functors

Notation. For $F : !A \to B$ and $a \in A$, define $\frac{\partial}{\partial a} F : !A \to B$ by

$$\left(\frac{\partial}{\partial a} F\right)(b, \alpha) = F(b, \alpha \otimes [a])$$

Proposition. The following hold:

$$\frac{\partial}{\partial a'} \frac{\partial}{\partial a} F \cong \frac{\partial}{\partial a} \frac{\partial}{\partial a'} F \qquad \frac{\partial}{\partial a}(F + G) \cong \frac{\partial}{\partial a}(F) + \frac{\partial}{\partial a}(G)$$

$$\frac{\partial}{\partial a}(F \cdot G) \cong \left(\frac{\partial}{\partial a} F\right) \cdot G + F \cdot \left(\frac{\partial}{\partial a} G\right)$$

$$\frac{\partial}{\partial a}(G \circ F) \cong \int^{b \in B} \left(\frac{\partial}{\partial b}(G)\right) \circ F \cdot \frac{\partial}{\partial a}(F)$$

Future work

- Coherence conditions for differential (cf. rewriting of differential λ-calculus)
- Extension to analytic functors between operad algebras
- Links to Taylor series expansion

Part of wider project on 2-dimensional models of linear logic, with M. Fiore, Z. Galal and F. Olimpieri.