Monoidal bicategories, differential linear logic, and analytic functors *

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Thanks to Pino for ...

- Showing me (and many others) that another mathematics is possible

- Crisp advice and constant support

Goal

Extend Joyal's theory of analytic functors to "many variables".

Based on jww: M. Fiore, R. Garner, M. Hyland, A. Joyal, C. Vasilakopoulou, G. Winskel and work of Z. Galal, A. Miranda, A. Slattery.

Plan

I. Review of Joyal's theory

II. Profunctors and Kleisli bicategories

III. Differentiation

+ Quiz

Symmetric sequences

Let $\quad \mathbb{P} = \coprod_{n \in \mathbb{N}} \Sigma_n$

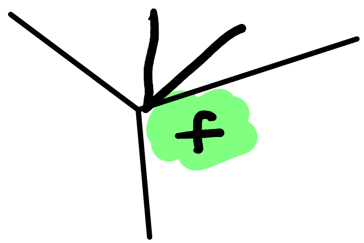
- **objects**: $n \in \mathbb{N}$
- **maps**: permutations

Def A symmetric sequence is a functor

$$F : \mathbb{P} \longrightarrow \mathrm{Set}$$

Idea

$f \in F[n] \iff$



Analytic functors

Given $F : \mathbb{P} \longrightarrow \underline{\mathrm{Set}}$, define

$$\begin{tikzcd}
\mathbb{P} \arrow[r] \arrow[dr, bend right, "F"'] & \underline{\mathrm{Set}} \arrow[d, "\widetilde{F}"] \arrow[dl, phantom, "\Rightarrow"] \\
& \underline{\mathrm{Set}}
\end{tikzcd}$$

$$\widetilde{F}(X) = \coprod_{n \in \mathbb{N}} \frac{F[n] \times X^n}{\Sigma_n}$$

$$\text{cf. } f(x) = \sum_{n \in \mathbb{N}} f_n \frac{x^n}{n!}$$

The calculus of analytic functors

$F + G$, $F \times G$, $F \boxtimes G$

[Maia & Mendez]

[Dwyer & Hess]

$G \circ F$, F'

substitution,

cf. [Kelly]

$$F'(X) = \sum_n \frac{F[n+1] \times X^n}{\sigma_n}$$

<u>Note</u> Links to symmetric operads.

Quiz

I am so fond of Pino that:

(A) I support a football team from Genoa.

(B) I go on holidays where he does

(C) My son is named like one of his children

Profunctors

Def. Let A, B be small categories.

A profunctor $F : A \nrightarrow B$ is a functor

$$F : B^{op} \times A \longrightarrow \mathbf{Set}$$

Prof = bicategory of

- **objects:** small categories
- **maps:** profunctors
- **2-cells:** nat. tr.

Note A proof is given in [FGHW 18]

Structure of Prof

- Symmetric monoidal : $A \otimes B = A \times B$
- Compact closed :

$$A^{\perp} = A^{op} \quad , \quad [A, B] = A^{\perp} \otimes B$$

- Coproducts (biproducts) :

$$A \oplus B = A + B$$

Note Part of a double category $\mathbb{P}\mathrm{rof}$.

A 2-monad on <u>Cat</u>

For $A \in \mathrm{Cat}$, $n \in \mathbb{N}$ let $?_n(A)$ be:

- **objects**: $(a_1, \dots, a_n)$, with $a_i \in A$ $1 \leq i \leq n$.
- **maps**: $(\sigma, f_1, \dots, f_n) : (a_1, \dots, a_n) \longrightarrow (a_1', \dots, a_n')$

  $\sigma \in \Sigma_n$, $f_i : a_i \longrightarrow a'_{\sigma(i)}$, $1 \leq i \leq n$.

$$?A = \coprod_{n \in \mathbb{N}} ?_n(A) = \text{free sym. mon. cat. on } A$$

$\Longrightarrow$ A 2-monad $? : \mathrm{Cat} \longrightarrow \mathrm{Cat}$

<u>Note</u> $?1 \cong \mathbb{P}$

A pseudomonad on Prof

Thm [FGHW 18] The 2-monad $?$: Cat $\longrightarrow$ Cat extends to a pseudomonad on Prof

Let $\mathrm{Prof}_?$ = Kleisli bicategory of $?$

- **objects**: small categories
- **maps**: $F : A \rightsquigarrow B$ $=_{\mathrm{def}}$ $F : A \nrightarrow ?B$
- **2-cells**: natural transformations.

Note For $A = B = 1$, $F : A \nrightarrow ?B \equiv F : \mathbb{P}^{op} \longrightarrow \mathrm{Set}$

Structure of $\mathbf{Prof}_?$

<u>Prop</u> $\mathbf{Prof}_?$ has coproducts.

<u>Thm</u> [GGV 24] $\mathbf{Prof}_?$ admits an oplax monoidal structure $\boxtimes$:

- $A \boxtimes B = A \times B$
- For $F_1 : A_1 \,⇸\, ?B_1$, $F_2 : A_2 \,⇸\, ?B_2$, have

$$A_1 \times A_2 \xrightarrow[\quad+\quad]{F_1 \times F_2} ?B_1 \times ?B_2 \xrightarrow[\quad+\quad]{\varphi_{B_1, B_2}} ?(B_1 \times B_2)$$

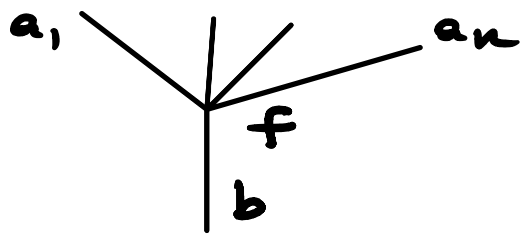
<u>Note</u> Extends [Maia & Mendez], [Dwyer & Hess]

Categorical symmetric sequences

Let **CatSym** $= (\mathrm{Prof}_?)^{op}$

- **objects:** small categories
- **maps:** $F : A \longrightarrow B \;=\; F : B \rightsquigarrow A \;=\; F : B \;\mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}}\; ?A \;=\; F : ?A^{op} \times B \longrightarrow \mathrm{Set}$

Idea

$f \in F[a_1, \dots, a_n ; b] \iff$ 

Analytic functors

For $F : A \longrightarrow B$ in CatSym, define

$$F : \overbrace{\underline{\mathrm{Set}}^{A}}^{\mathbb{R}^n} \longrightarrow \overbrace{\underline{\mathrm{Set}}^{B}}^{\mathbb{R}^m}$$

$$(X_a)_{a \in A} \longmapsto \left(\int^{\alpha \in ?A} F[\alpha ; b] \times \underbrace{X^{\alpha}}_{X(a_1) \times \dots \times X(a_n)} \right)_{b \in B}$$

Remark When $A = B = 1$, get back Joyal's analytic functors.

Cartesian closed structure

Obs CatSym has finite products,

as Prof? has finite coproducts

Thm [FGHW 08, GJ 17]

CatSym is cartesian closed.

Next An explanation

Duality

We have $A \mapsto A^{\perp} =_{\text{def}} A^{op}$ on **Prof**, so

$$\begin{tikzcd}
\mathbf{Prof} \arrow[r, "(-)^{\perp}"] \arrow[rrr, bend right, "!"'] & \mathbf{Prof} \arrow[r, "?"] & \mathbf{Prof} \arrow[r, "(-)^{\perp}"] & \underline{\mathbf{Prof}}
\end{tikzcd}$$

$\sim$ a pseudo<u>co</u>monad

<u>Note</u> Some kind of model of Linear Logic

Cf. [Cattani & Winskel '04]

A linear exponential pseudocomonad

Thm [FGH 24] $!(-) : \mathrm{Prof} \longrightarrow \mathrm{Prof}$

satisfies the axioms for a linear exponential pseudocomonad.

Note: we need

differently from $?(-)$

- $!(-)$ sym. monoidal pseudocomonad
- $\underbrace{\otimes \text{ on pseudo-algebras}}_{\text{[Miranda '24]}}$ is cartesian.

The cokleisli bicategory

Cor $\mathbf{Prof}_!$ = cokleisli bicategory of $!$

- **objects**: small categories
- **maps**: $F : A \rightsquigarrow B \quad =_{def} \quad F : !A \,⇸\, B$
- **2-cells**: natural transformations.

is cartesian closed, with

$$A \,\&\, B \;=_{def}\; A \oplus B \qquad\qquad A \Rightarrow B = [\,!A, B\,]$$

Note $\mathbf{CatSym} \simeq \mathbf{Prof}_!$

Quiz

I once invited Pino to a workshop in Formal Topology in Palermo at short notice. He:

(A) Put the phone down

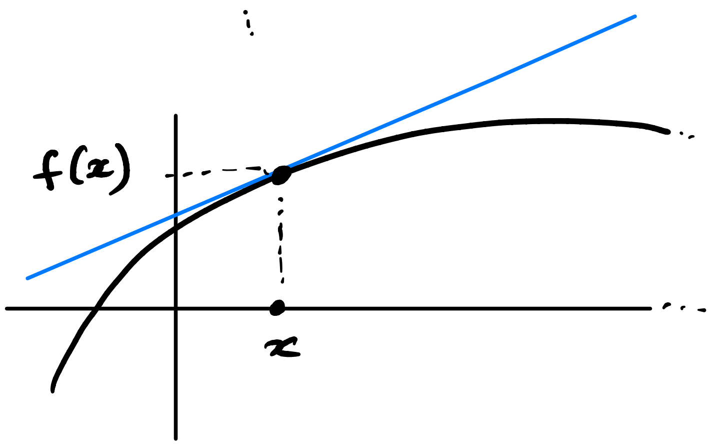
(B) Enthusiastically accepted

(C) Told me what he thinks of Formal Topology.

Towards differentiation

Recall For $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $\vec{x} \in \mathbb{R}^m$,

the Jacobian $Df_{\vec{x}}$ is an $(m \times n)$-matrix



Then $\mathbf{f}'(\mathbf{x})\mathbf{e}_j$ is the jth column vector of $[\mathbf{f}'(\mathbf{x})]$, and (27) shows therefore that the number $(D_j f_i)(\mathbf{x})$ occupies the spot in the ith row and jth column of $[\mathbf{f}'(\mathbf{x})]$. Thus

$$[\mathbf{f}'(\mathbf{x})] = \begin{bmatrix} (D_1 f_1)(\mathbf{x}) & \cdots & (D_n f_1)(\mathbf{x}) \\ \cdots & \cdots & \cdots \\ (D_1 f_m)(\mathbf{x}) & \cdots & (D_n f_m)(\mathbf{x}) \end{bmatrix}.$$

If $\mathbf{h} = \Sigma h_j \mathbf{e}_j$ is any vector in R^n, then (27) implies that

$$\mathbf{f}'(\mathbf{x})\mathbf{h} = \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n} (D_j f_i)(\mathbf{x}) h_j \right\} \mathbf{u}_i . \tag{30}$$

$\Longrightarrow$ $Df_{\vec{x}}(\vec{a})$ is

- linear in $\vec{a} \in \mathbb{R}^m$
- non-linear in $\vec{x} \in \mathbb{R}^m$

Want An operator $D[-]$, taking

$$F : \,!A \longrightarrow B$$

to

$$D[F] : \underbrace{A}_{\text{linear}} \otimes \underbrace{!A}_{\text{non-linear}} \longrightarrow B$$

subject to usual rules of differential calculus (constant, linear, sum, product, chain, ...)

Codereliction

Fact [Fiore] In Prof, it is sufficient to have $\bar{d}_A : A \nrightarrow\, !A$ satisfying

$$\begin{tikzcd}
A \arrow[r, "\bar{d}"] \arrow[dr, bend right, "1_A"'] & !A \arrow[d, "d"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes I \arrow[r, "\cong"] \arrow[d, "\bar{d} \otimes \bar{w}"'] & A \arrow[r, "d"] & !A \arrow[d, "p"] \\
!A \otimes !A \arrow[r, "\bar{d} \otimes p"'] & !!A \otimes !!A \arrow[r, "\bar{c}"'] & !!A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes !B \arrow[r, "\bar{d} \otimes 1"] \arrow[d, "1 \otimes d"'] & !A \otimes !B \arrow[d, "m^2"] \\
A \otimes B \arrow[r, "\bar{d}"'] & !(A \otimes B)
\end{tikzcd}$$

Note For $F : !A \nrightarrow B$, get $D[F] : A \otimes !A \nrightarrow B$ as

$$A \otimes !A \xrightarrow[\quad]{\bar{d} \otimes 1} !A \otimes !A \xrightarrow{\bar{c}} !A \xrightarrow{F} B$$

Differential structure on Prof!

Define $\bar{d}_A : A \nrightarrow\, !A$ by

$$\begin{aligned} !A^{op} \times A &\xrightarrow{\bar{d}_A} \mathbf{Set} \\ \alpha, a &\longmapsto\, !A[\alpha, (a)] \end{aligned}$$

Thm [FGH '24] $\bar{d}$ satisfies the axioms for codereliction in a bicategorical sense, canonically.

Differentiation

For $F : \mathbf{Set}^A \longrightarrow \mathbf{Set}^B$ analytic, $X \in \mathbf{Set}^A$

the Jacobian $dF(X) : A^{op} \times B \to \mathbf{Set}$ is given by

$$dF(X)(a,b) = \sum_{\vec{a} \in !A} F[\underbrace{\vec{a} \oplus (a)}_{\substack{\text{length } n+1 \\ \text{if } \vec{a} \text{ has length } n.}} ; b] \times X^{\vec{a}}$$

Corollary All the rules of calculus hold for analytic functors.

Other work

- [Galal 24] has shown Prof$_!$ admits fix points in a bicategorical sense, cf. [Plotkin & Simpson]
- [GJ 17] also constructs from CatSym a bicategory of sym multicategories and bimodules, like Prof, and shows it is cartesian closed.
- ...

Work in progress

with R. Garner and C. Vasilakopoulou

Use $\boxtimes$ to extend the Boardman–Vogt tensor product of symmetric multicategories to their bimodues

cf. [Garner & Lopez Franco], [Dwyer & Hess].

References

[FGHW 08] The cartesian closed bicategory of generalised species of structures, J LMS

[GJ 17] On operads, bimodules and analytic functors, Memoirs AMS

[FGHW 18] Relative pseudomonads, Kleisli bicategories and ... Selecta Math.

[GGV 24] Monoidal Kleisli bicategories and the arithmetic ... Doc. Math.