# Monoidal bicategories, differential linear logic, and analytic functors *

N. Gambino

jww M. Fiore and M. Hyland

BLC 2024

Birmingham, 7/9/2024

* ArXiv, 2024.

# What do logicians have in common? *

**Theorem** (Completeness for propositional logic)

$\Gamma \vdash A$ iff $\Gamma \models A$

$\Gamma \vdash A$: defined syntactically via calculus

- Hilbert system
- Natural deduction
- Sequent calculus

$\Gamma \models A$: defined semantically via truth values

---

* Alex Wilkie, LC2010.

# Deduction rules

$$\dfrac{}{A \vdash A}\,ax \qquad \dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge_R \qquad \dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,\wedge_L$$

**Exercise** Prove $A \vdash A \wedge A$

**Solution**

$$\dfrac{\dfrac{\dfrac{}{A \vdash A} \qquad \dfrac{}{A \vdash A}}{A, A \vdash A \wedge A}}{A \vdash A \wedge A}\,c$$

**Exercise** Prove $A \wedge B \vdash A$

**Solution**

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, B \vdash A}\,w}{A \wedge B \vdash A}$$

# Linear Logic

- Drop structural rules for arbitrary formulas.

$$\frac{}{A \vdash A}\,ax \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,\otimes_R \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,\otimes_L$$

- Note $A \nvdash A \otimes A$, $A \otimes B \nvdash A$

- We think of hypotheses as resources

- This is the difference between

$(\mathbb{L}, \times, 1)$ cartesian, e.g. Set

$(\mathbb{L}, \otimes, I)$ monoidal, e.g. $\mathrm{Vect}_k$

# Proofs-as-maps

## Linear Logic

$$\frac{}{A \vdash A}\,ax$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,\otimes_R$$

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,cut$$

## Monoidal category

$$A \xrightarrow{id_A} A$$

$$\frac{\Gamma \xrightarrow{a} A \qquad \Delta \xrightarrow{b} B}{\Gamma \otimes \Delta \xrightarrow{a \otimes b} A \otimes B}$$

$$\frac{\Gamma \xrightarrow{a} A \qquad \Delta \otimes A \xrightarrow{b} B}{\Gamma \otimes \Delta \xrightarrow{a \otimes 1} A \otimes \Delta \xrightarrow{b'} B}$$

# Commutative diagrams as proof equations

Functoriality of $\otimes$ corresponds to:

$$\dfrac{\dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \dfrac{A \vdash A' \qquad B \vdash B'}{A \otimes B \vdash A' \otimes B'}}{\Gamma, \Delta \vdash A' \otimes B'}\,\text{cut}$$

$$=$$

$$\dfrac{\dfrac{\Gamma \vdash A \qquad A \vdash A'}{\Gamma \vdash A'}\,\text{cut} \qquad \dfrac{\Delta \vdash B \qquad B \vdash B'}{\Delta \vdash B'}\,\text{cut}}{\Gamma, \Delta \vdash A' \otimes B'}$$

# Linear Logic

**Idea** : structural rules for some formulas, via **modality** :

"of course A" $\rightsquigarrow$ $!A$ $\rightsquigarrow$ we can use $!A$ as many times as we want
"bang A"

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ d \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ p$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ w \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ c$$

**Note** $!A \vdash !A \otimes !A$ , $!A \otimes !B \vdash !A$

# Linear exponential comonad

$(\mathbb{L}, \otimes, I)$ a symmetric monoidal category with

- $\mathbb{L} \xrightarrow{!} \mathbb{L}$

- maps $d_A : !A \longrightarrow A$
- maps $w_A : !A \longrightarrow I$
- maps $p_A : !A \longrightarrow !!A$
- maps $c_A : !A \longrightarrow !A \otimes !A$

**KEY IDEA**

- maps $f : A \longrightarrow B$ in $\mathbb{L}$ are linear
- maps $f : !A \longrightarrow B$ in $\mathbb{L}$ are non-linear

# Differential Linear Logic (T. Ehrhard & L. Reignier)

Maps in many models of Linear Logic admit

a well-behaved differentiation operation

$\Rightarrow$ introduce a syntax for it

$\Rightarrow$ differential $\lambda$-calculus, differential linear logic

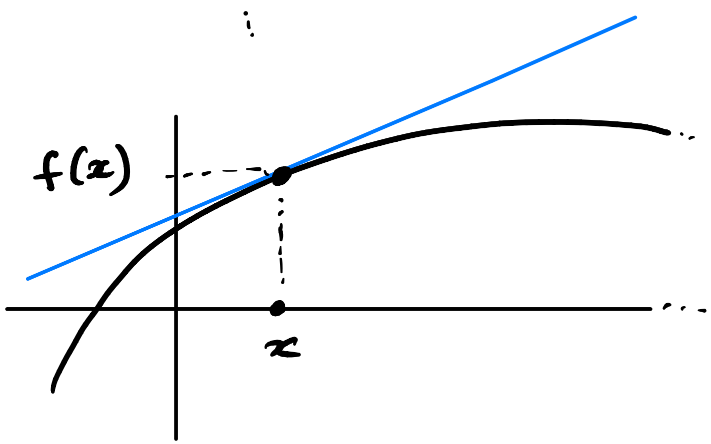
$\Rightarrow$ applications to ordinary $\lambda$-calculus

(via Taylor series)

# 2. Differential Linear Logic

**Recall** For $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $\vec{x} \in \mathbb{R}^m$,

the Jacobian $Df_{\vec{x}}$ is an $(m \times n)$-matrix




Then $\mathbf{f}'(\mathbf{x})\mathbf{e}_j$ is the $j$th column vector of $[\mathbf{f}'(\mathbf{x})]$, and (27) shows therefore that the number $(D_j f_i)(\mathbf{x})$ occupies the spot in the $i$th row and $j$th column of $[\mathbf{f}'(\mathbf{x})]$. Thus

$$[\mathbf{f}'(\mathbf{x})] = \begin{bmatrix} (D_1 f_1)(\mathbf{x}) & \cdots & (D_n f_1)(\mathbf{x}) \\ \cdots & \cdots & \cdots \\ (D_1 f_m)(\mathbf{x}) & \cdots & (D_n f_m)(\mathbf{x}) \end{bmatrix}.$$

If $\mathbf{h} = \Sigma h_j \mathbf{e}_j$ is any vector in $R^n$, then (27) implies that

$$\mathbf{f}'(\mathbf{x})\mathbf{h} = \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n} (D_j f_i)(\mathbf{x}) h_j \right\} \mathbf{u}_i. \tag{30}$$

$\Longrightarrow$ $Df_{\vec{x}}(\vec{a})$ is
- linear in $\vec{a} \in \mathbb{R}^m$
- non-linear in $\vec{x} \in \mathbb{R}^m$

# Differential categories (Blute, Cockett, Seely)

Let $(\mathbb{L}, \otimes, I)$ with $(!, p, d)$ as before.

**WANT** An operator $D[-]$

$$f : \,!A \longrightarrow B$$

to

$$D[f] : \overset{\text{linear}}{A} \otimes \overset{\text{non-linear}}{!A} \longrightarrow B$$

subject to usual axioms of differential calculus:

constant, sum, product, chain.

## Theorem (Fiore, Blute & Cockett & Lemay & Seely)

A differential category is determined by maps

$$\bar{d}_A : A \longrightarrow \,!A \quad \text{satisfying}$$

$$\begin{tikzcd}
A \arrow[r, "\bar{d}"] \arrow[dr, bend right, "1_A"'] & !A \arrow[d, "d"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes I \arrow[r, "\cong"] \arrow[d, "\bar{d} \otimes \bar{w}"'] & A \arrow[r, "d"] & !A \arrow[d, "p"] \\
!A \otimes !A \arrow[r, "\bar{d} \otimes p"'] & !!A \otimes !!A \arrow[r, "\bar{c}"'] & !!A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes !B \arrow[r, "\bar{d} \otimes 1"] \arrow[d, "1 \otimes d"'] & !A \otimes !B \arrow[d, "m^2"] \\
A \otimes B \arrow[r, "\bar{d}"'] & !(A \otimes B)
\end{tikzcd}$$

## Note

For $f : !A \longrightarrow B$, get $D[f] : A \otimes !A \longrightarrow B$ as

$$A \otimes !A \xrightarrow{\bar{d} \otimes 1} !A \otimes !A \xrightarrow{\bar{c}} !A \xrightarrow{f} B$$

# 3. Analytic functors

| Calculus | Functors |
| --- | --- |
| $\mathbb{R} \xrightarrow{f} \mathbb{R}$ linear | $\underline{Set} \xrightarrow{F} \underline{Set}$ linear functor |
| $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ linear | $\underline{Set}^{A^{op}} \longrightarrow \underline{Set}^{B^{op}}$ linear functor |
| $\mathbb{R} \xrightarrow{f} \mathbb{R}$ | $\underline{Set} \xrightarrow{F} \underline{Set}$ analytic (Joyal '86) |
| $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ | $\underline{Set}^{A^{op}} \longrightarrow \underline{Set}^{B^{op}}$ analytic (FGHW '08) |

# Linear maps and Linear Functors

**Definition** Let $A$, $B$ be small categories. A **linear map*** $M : A \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} B$ is a functor

$$\begin{array}{ccc} B^{op} \times A & \xrightarrow{M} & \underline{Set} \\ (b, a) & \longmapsto & M(b,a) \end{array}$$

~ the $(b,a)$-entry of the matrix

The **linear functor** $\tilde{M} : \underline{Set}^{A^{op}} \longrightarrow \underline{Set}^{B^{op}}$

$$(\tilde{M} x)_b = \sum_{a \in A} M(b,a) \times x_a \Big/ \underline{\sim}$$

a quotient

* Bénabou's profunctors.

# Towards Linear Logic

$\mathbb{L}$

**Lin** = the category of

- small categories
- matrices

**Prop** The category **Lin** admits a symmetric monoidal structure where $A \otimes B =_{\text{def}} A \times B$.

( It has also biproducts and is compact closed )

# The linear exponential comonad

For $A \in \underline{\mathrm{Lin}}$, define $!A \in \underline{\mathrm{Lin}}$ as the free symmetric monoidal category on $A$:

- **objects**: $\vec{a} = (a_1, \ldots, a_n)$, for $n \in \mathbb{N}$.
- **maps**: $(\sigma, \vec{f}) : (a_1, \ldots, a_n) \longrightarrow (a_1', \ldots, a_n')$

  where $\sigma \in \Sigma_n$, $f_i : a_i \longrightarrow a'_{\sigma(i)}$

**Theorem** [FGH] This definition extends to a linear exponential comonad on $\underline{\mathrm{Lin}}$.

# Analytic functors

**Definition** Let $A, B$ be small categories. A **non-linear map** $F : A \longrightarrow B$ is a linear map $F : {!A} \longrightarrow\!\!\!\!\!\!+\;\, B$, ie. a functor $F : B \times {!A}^{op} \longrightarrow \underline{\underline{Set}}$

The **analytic functor** $\tilde{F} : \underline{Set}^{A^{op}} \longrightarrow \underline{Set}^{B^{op}}$ is

$$(\tilde{F} X)_b = \sum_{\vec{a} \in !A} F[b; \vec{a}] \times X^{\vec{a}} \Big/ \simeq$$

quotient

$=_{def} X(a_1) \times \ldots \times X(a_n)$

"coefficients"

# Joyal's analytic functors

Let $A = B = 1$. Then $!1 = \mathbb{P} =$ category of

- natural numbers
- permutations

$$F : \,!1 \longrightarrow\!\!\!\!\!\!\mapsto 1 \quad = \quad F : \mathbb{P} \longrightarrow \underline{\mathrm{Set}}$$ and

$$\tilde{F} : \underline{\mathrm{Set}} \longrightarrow \underline{\mathrm{Set}}$$

$$X \longmapsto \sum_{n \in \mathbb{N}} F[n] \times X^n / \simeq_{\sigma_n}$$

**Note** Cf. $f(x) = \sum_{n \in \mathbb{N}} f_n \frac{x^n}{n!}$.

# Differentiation

- For $f(x) = \sum_n f_n \frac{x^n}{n!}$ , $f'(x) = \sum_n f_{n+1} \frac{x^n}{n!}$

- For $F : \underline{\mathrm{Set}} \longrightarrow \underline{\mathrm{Set}}$

$$F'(X) = \sum_{n \in \mathbb{N}} F[n+1] \times X^n / \simeq_{\sigma_n} \qquad \text{(Joyal)}$$

- For $F : \underline{\mathrm{Set}}^{A^{op}} \longrightarrow \underline{\mathrm{Set}}^{B^{op}}$

What should we do ?

# Differentiation

Define $\overline{d}_A : A \longrightarrow\!\!\!\!\!\!\!\!+\quad !A$ by

$$\begin{array}{ccc} !A \times A^{op} & \xrightarrow{\overline{d}_A} & \underline{\mathbf{Set}} \\ (\vec{a}, a) & \longmapsto & !A[\vec{a}, (a)] \end{array}$$

**Theorem [FGH]** This satisfies the axioms for a differential category.

# Differentiation

For $F : \underline{\mathrm{Set}}^{A^{op}} \longrightarrow \underline{\mathrm{Set}}^{B^{op}}$ analytic, $X \in \underline{\mathrm{Set}}^{A^{op}}$

the Jacobian $dF(X) : A \longrightarrow\!\!\!\!\!\!\!\!+\;\; B$ is given by

$$dF(X)(b,a) = \sum_{\vec{a} \in !A} F[b; \underbrace{\vec{a} \oplus (a)}_{\substack{\text{length } n+1 \\ \text{if } \vec{a} \text{ has length } n.}}] \times X^{\vec{a}}$$

Corollary All the rules of calculus hold

for analytic functors.

# Monoidal bicategories

The example of analytic functors does not fit into the theory of monoidal categories since it is inherently 2-categorical, e.g.

$$\begin{tikzcd}
& & ((A\otimes B)\otimes C)\otimes D \arrow[dll, "a"'] \arrow[drr, "a"] & & \\
(A\otimes(B\otimes C))\otimes D \arrow[dr, "a"'] \arrow[rrrr, phantom, "\Downarrow\pi"] & & & & (A\otimes B)\otimes(C\otimes D) \arrow[dl, "a"] \\
& A\otimes((B\otimes C)\otimes D) \arrow[rr, "a"'] & & (A\otimes(B\otimes(C\otimes D))) &
\end{tikzcd}$$

$\Rightarrow$ The paper develops the theory of monoidal bicategories (Kapranov-Voevodsky, ... )

# Coherence and rewriting

$$\dfrac{\dfrac{!A \xrightarrow{f} B}{!A \xrightarrow{f^*} !B} \qquad \dfrac{!B \xrightarrow{d_B} B}{!B \xrightarrow{d_B^*} !B}}{!A \xrightarrow{d_B^* f^*} !B}\ \text{cut}$$

$\swarrow$ $\Downarrow$

$$\dfrac{\dfrac{\dfrac{!A \xrightarrow{f} B}{!A \xrightarrow{f^*} !B} \qquad !B \xrightarrow{d_B} B}{!A \xrightarrow{d_B f^*} B}\ \text{cut}}{!A \xrightarrow{(d_B f^*)^*} !B}$$

$$\dfrac{\dfrac{!A \xrightarrow{f} B}{!A \xrightarrow{f^*} !B} \qquad !B \xrightarrow{1} !B}{!A \xrightarrow{f^*} !B}\ \text{cut}$$

$\searrow$ $\swarrow \cong$

$$\dfrac{!A \xrightarrow{f} B}{!A \xrightarrow{f^*} !B}$$

# Summary

- The bicategory <u>Lin</u> of small categories and linear maps can be equipped with the structure of a model of differential linear logic.

- This gives an extension of Joyal's differential calculus for analytic functors to 'many variables'.