Algebraic Number Theory 2019-20
Example Sheet 5

There are no questions to hand in! Solutions will be provided early in term 2. If you want to ask questions about this example sheet, the best way is on Moodle.

1. Find the fundamental units of the following quadratic fields.
$$\mathbb{Q}(\sqrt{3}), \quad \mathbb{Q}(\sqrt{13})$$

2. Find all solutions in integers of the following equations:
   (a) $x^2 - 21y^2 = 1$
   (b) $x^2 - 26y^2 = 1$
   (c) $x^2 - 13y^2 = 1$
   (d) $x^2 - 11y^2 = 1$
   In each case, you should prove that the formula you write down does indeed give integer solutions to the equation. For (b) and (d), you should prove that you have found all the integer solutions. For (a) and (c), you can quote a fact from the end of the lecture notes to guarantee that you have found all the integer solutions, but don't need to fully prove it.

Harder questions (non-examinable)

3. Let ε be the fundamental unit of $K = \mathbb{Q}(\sqrt{26})$. Prove that all solutions in integers of the equation $x^2 - 26y^2 = \pm 10$ are given by
$$x + y\sqrt{26} = \pm\varepsilon^{2n}(\varepsilon + 1) \text{ or } \pm\varepsilon^{2n+1}(\varepsilon - 1),$$
for some $n \in \mathbb{Z}$. (Hint: prove that $\langle \varepsilon + 1 \rangle$ and $\langle \varepsilon - 1 \rangle$ are the only ideals of norm 10 in $\mathcal{O}_K$. You might find it helpful to use the information about prime ideals of $\mathcal{O}_K$ from example sheet 4, Q4.)

4. Find a number field whose unit group is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}$.

5. Let $K = \mathbb{Q}(\sqrt[3]{2})$. You may suppose that $1, \sqrt[3]{2}, \sqrt[3]{2}^2$ is an integral basis for $\mathcal{O}_K$. Show that
$$U(K) = \{\pm(\sqrt[3]{2} - 1)^n \;:\; n \in \mathbb{Z}\}.$$
You may need to use `WolframAlpha`, `MATLAB` or a similar package to compute approximations to the embeddings of some algebraic numbers.

If you want more practice, you could try finding the class group for all quadratic fields $\mathbb{Q}(\sqrt{\pm d})$ and the fundamental unit for all real quadratic fields $\mathbb{Q}(\sqrt{d})$ for $d \leq 30$ (or as many of them as you feel like!) The class groups should all be do-able by hand, but some of the fundamental units are large and you might want to use a computer.