Algebraic Number Theory 2019-20
Example Sheet 1

Hand in the answers to questions 6, 8, 11 (marked with †).
Deadline 12 noon Monday, week 4 (21 October)
For questions about the example sheet, it is best to ask them on Moodle. Questions must be asked before 5 pm on Friday to get an answer before the deadline.

1. Find the minimal polynomials over $\mathbb{Q}$ of the following algebraic numbers:

$$\sqrt{-5}, \quad \sqrt{2}+\sqrt{7}, \quad \exp(2\pi i/5), \quad \cos(2\pi/7), \quad \frac{\sqrt[3]{3}}{2}, \quad \exp(2\pi i/3)+2, \quad \frac{\sqrt{3}+\sqrt{5}}{2}.$$

2. Find the minimal polynomial of $\frac{1+i}{\sqrt{2}}$ over the following fields:

$$\mathbb{Q}, \quad \mathbb{Q}(i), \quad \mathbb{Q}(\sqrt{2}), \quad \mathbb{Q}(\sqrt{-2}).$$

3. What is the degree of the following extensions?

$$\mathbb{Q}(\sqrt{5},\sqrt{7},\sqrt{35})/\mathbb{Q}, \quad \mathbb{Q}(\sqrt[3]{2},\sqrt{2})/\mathbb{Q}.$$

4. You probably know that e and π are transcendental. Show that e and π are algebraic over the field $\mathbb{Q}(e+\pi, e\pi)$.

5. (i) Let K be a number field of degree 2. Prove that $K=\mathbb{Q}(\sqrt{d})$ for some square-free integer $d \neq 1$.
 (ii) Let d_1, d_2 be square-free integers $\neq 1$, with $d_1 \neq d_2$. Prove that
 (a) $\mathbb{Q}(\sqrt{d_1}) \neq \mathbb{Q}(\sqrt{d_2})$,
 (b) $\mathbb{Q}(\sqrt{d_1},\sqrt{d_2}) = \mathbb{Q}(\sqrt{d_1}+\sqrt{d_2})$.

†6. Let K be a number field and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be a $\mathbb{Q}$-basis for K. Prove that $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_r)$.

7. Let p be a prime number and let $\zeta_p = \exp(2\pi i/p)$. Prove that the minimal polynomial of ζ_p is $X^{p-1}+X^{p-2}+\cdots+X+1$ and deduce that $[\mathbb{Q}(\zeta_p):\mathbb{Q}] = p-1$. Write down a $\mathbb{Q}$-basis for $\mathbb{Q}(\zeta_p)$.

†8. (i) Show that $f(X) = X^3-4X+2$ is irreducible over $\mathbb{Q}$.
 (ii) By considering $f(0)$ and $f(1)$, or otherwise, show that f has 3 real roots.
 (iii) Let α be a root of $f(X)$. What are the degree and signature of $\mathbb{Q}(\alpha)$?

9. Let d be a non-cube rational number. Describe the embeddings of $\mathbb{Q}(\sqrt[3]{d})$. What is the signature of $\mathbb{Q}(\sqrt[3]{d})$? Conclude that the field $\mathbb{Q}(\alpha)$ from question 8 is not equal to $\mathbb{Q}(\sqrt[3]{d})$ for any $d \in \mathbb{Q}$.

10. Let $\sigma : \mathbb{Q}(\sqrt{5}) \hookrightarrow \mathbb{C}$ be given by $\sigma(a+b\sqrt{5}) = a-b\sqrt{5}$. Explicitly write down the embeddings $\tau : \mathbb{Q}(\sqrt{5},\sqrt{6}) \hookrightarrow \mathbb{C}$ that extend σ.

†11. (i) Let $K = \mathbb{Q}(\sqrt{2})$ and let $\alpha = 1 + \sqrt{2}$. Prove that the embeddings of K can be labelled as σ_1, σ_2 in such a way that $\sigma_1(\alpha) > 0$ and $\sigma_2(\alpha) < 0$.

(ii) Let $\beta = \sqrt{\alpha}$ and $L = \mathbb{Q}(\beta)$. Prove that $\beta \notin K$. (Hint: suppose for contradiction that $\beta \in K$ and consider $\sigma_2(\beta)$.)

(iii) Describe the embeddings of L extending σ_1, and the embeddings of L extending σ_2. Which are real and which are complex? What is the signature of L?

(iv) Write down a number field of signature $(4, 0)$. Explain in around three or four sentences why your field has this signature. (Do not write out a full proof – marks will be awarded for explaining the important steps briefly.)

Correction: (ii) originally said $L = \mathbb{Q}(\sqrt{\beta})$ but it should be $L = \mathbb{Q}(\beta)$. (iii) originally said "embeddings of L describing σ_2" instead of "extending σ_2."

12. A field K is **algebraically closed** if every β that is algebraic over K belongs to K.

(i) Explain why $\mathbb{C}$ is algebraically closed.

(ii) Show that $\overline{\mathbb{Q}}$ is algebraically closed.

13. (i) Let $L = K(\sqrt{d})$ where d is a non-square in K. Prove that, if $x \in L$ and $x^2 \in K$, then either $x \in K$ or $x\sqrt{d} \in K$.

(ii) Let $K_r = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_r})$ where $d_1, \ldots, d_r$ are square-free integers. Prove by induction that, if $x \in K_r$ and $x^2 \in \mathbb{Q}$, then

$$x = q\sqrt{d_1^{a_1} d_2^{a_2} \cdots d_r^{a_r}}$$

where $q \in \mathbb{Q}$ and $a_1, \ldots, a_r = 0$ or 1.

(iii) Suppose that $d_1, \ldots, d_r$ are distinct prime numbers. Determine the degree $[\mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_r}) : \mathbb{Q}]$.