Algebraic Number Theory 2019
Example Sheet 4

Hand in the answers to questions 6, 9 (marked with †).
Deadline 2pm Friday, Week 10.

1. Let $K = \mathbb{Q}(\sqrt{-2})$. Show that $\mathcal{O}_K$ is a principal ideal domain. Deduce that every prime $p \equiv 1,\ 3 \pmod 8$ can be written as $p = x^2 + 2y^2$ with $x,\ y \in \mathbb{Z}$. (You will need to use quadratic reciprocity from Introduction to Number Theory.)

2. Compute the class groups of the following quadratic fields.
$$\mathbb{Q}(\sqrt{5}), \quad \mathbb{Q}(\sqrt{-6}), \quad \mathbb{Q}(\sqrt{7}).$$

3. Prove that the class group of $\mathbb{Q}(\sqrt{-30})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

4. Show that the cyclotomic field $\mathbb{Q}(\zeta_5)$ has class number 1.

5. Let $\mathfrak{a}, \mathfrak{b}$ be ideals of $\mathcal{O}_K$ such that there is no prime ideal which divides both $\mathfrak{a}$ and $\mathfrak{b}$. Suppose that
$$\mathfrak{a}\mathfrak{b} = \mathfrak{c}^n$$
for some ideal $\mathfrak{c} \subseteq \mathcal{O}_K$ and some positive integer n. Prove that there are ideals $\mathfrak{a}', \mathfrak{b}' \subseteq \mathcal{O}_K$ such that
$$\mathfrak{a} = (\mathfrak{a}')^n, \quad \mathfrak{b} = (\mathfrak{b}')^n.$$

†6. (i) Prove that the ring of integers of $\mathbb{Q}(\sqrt{-11})$ is a PID.
(ii) Prove that if $x, y \in \mathbb{Z}$ satisfy $x^3 = y^2 + 11$, then there exist $u, v \in \mathbb{Z}$ such that
$$\left(\frac{u + v\sqrt{-11}}{2}\right)^3 = y + \sqrt{-11}.$$
(iii) Show that the equation $x^3 = y^2 + 11$ has exactly four solutions in rational integers. Verify that two of these solutions are $(15, \pm 58)$; find the other two.

7. Prove that the only integer solutions to the equation $x^3 = y^2 + 2$ are $(3, \pm 5)$.

8. Determine the fundamental unit of the following quadratic fields:
$$\mathbb{Q}(\sqrt{3}), \quad \mathbb{Q}(\sqrt{13}).$$

†9. Let $K = \mathbb{Q}(\sqrt{10})$.
(i) Determine the fundamental unit of K.
(ii) Prove that K contains a unique prime ideal $\mathfrak{p}_2$ of norm 2, and that $\mathfrak{p}_2$ is not principal. (For the second part, you may find it helpful to work mod 5.)
(iii) Show that $\mathrm{Cl}(K) \cong \mathbb{Z}/2\mathbb{Z}$. (You may find it useful to consider the factorisation of the ideal $\langle 2 + \sqrt{10} \rangle$, which has norm 6.)

10. Let $K = \mathbb{Q}(\sqrt{26})$.
    (i) Determine the fundamental unit ε of K.
    (ii) Show that $\mathrm{Cl}(K) \cong \mathbb{Z}/2\mathbb{Z}$. (Note that $\mathrm{Nm}_{K/\mathbb{Q}}(\varepsilon + 1) = 10$.)
    (iii) Prove that all solutions in integers of the equation $x^2 - 26y^2 = \pm 10$ are given by
$$x + y\sqrt{26} = \pm\varepsilon^n(\varepsilon \pm 1).$$

11. Find a number field whose unit group is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}$.

12. Let $K = \mathbb{Q}(\sqrt[3]{2})$. You may suppose that $1, \sqrt[3]{2}, \sqrt[3]{2}^2$ is an integral basis for $\mathcal{O}_K$. Show that
$$U(K) = \{\pm(\sqrt[3]{2} - 1)^n \ : \ n \in \mathbb{Z}\}.$$
You may need to use `WolframAlpha`, `MATLAB` or a similar package to compute approximations to the embeddings of some algebraic numbers.