ALGEBRAIC NUMBER THEORY 2019
EXAMPLE SHEET 1

Hand in the answers to questions 3, 5, 6 (marked with †).
Deadline 2pm Friday, Week 3.

1. Which of the following are algebraic numbers and which are algebraic integers [1]:

$$23/7, \quad \sqrt{-5}, \quad \sqrt{2}+\sqrt{7}, \quad \exp(2\pi i/5), \quad \cos(2\pi/7), \quad \frac{\sqrt[3]{3}}{2}.$$

Write down their minimal polynomials, conjugates [2], norms and traces.

2. You probably know that e and π are transcendental. Show that e and π are algebraic over the field $\mathbb{Q}(e+\pi, e\cdot\pi)$.

†3. Suppose d_1, d_2 are squarefree integers $\neq 0$, 1.
(i) Show $\mathbb{Q}(\sqrt{d_1}) = \mathbb{Q}(\sqrt{d_2})$ if and only if $d_1 = d_2$.
(ii) Suppose $d_1 \neq d_2$ and let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$.
- Write down a basis for $K/\mathbb{Q}$.
- Show that $K = \mathbb{Q}(\sqrt{d_1}+\sqrt{d_2})$.

4. Let p be a prime number and let $\zeta_p = \exp(2\pi i/p)$. Prove that the minimal polynomial of ζ_p is $X^{p-1}+X^{p-2}+\cdots+X+1$ and deduce that $[\mathbb{Q}(\zeta_p):\mathbb{Q}] = p-1$. Write down a $\mathbb{Q}$-basis for $\mathbb{Q}(\zeta_p)$.

†5. Again let p be a prime number and $\zeta_p = \exp(2\pi i/p)$. Let $K = \mathbb{Q}(\zeta_p)$ and $\alpha = 1+\zeta_p$. Calculate $\mathrm{Nm}_{K/\mathbb{Q}}(\alpha)$ by using the minimal polynomial of α.

†6. Let $f(X) = X^3 - 2X - 2$.
(i) Show that f is irreducible.
(ii) Let θ be a root of f. Find the minimal polynomial for $1+\theta+\theta^2$.
Hint: write the matrix M_θ for m_θ in terms of the basis $1, \theta, \theta^2$. Argue that $M_{1+\theta+\theta^2} = I + M_\theta + M_\theta^2$, deduce the characteristic polynomial and minimal polynomial.

7. This exercise is a revision of the Cayley–Hamilton theorem, from the perspective of algebraic number theory. Let $f(X) = a_0 + a_1X + \cdots + a_{n-1}X^{n-1} + X^n$ be a monic polynomial. We define the **companion matrix** of f to be

$$C_f = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

[1] An algebraic number α is called an **algebraic integer** if its minimal polynomial μ_α belongs to $\mathbb{Z}[X]$.

[2] Two algebraic numbers α, β are **conjugates** if they share the same minimal polynomial.

(i) Use induction on the degree n to show that the characteristic polynomial of C_f is f.
Hint: For the inductive step write $g = a_1 + a_2X + \cdots + a_{n-1}X^{n-2} + X^{n-1}$ and show that $\det(XI_n - C_f) = X \cdot \det(XI_{n-1} - C_g) + a_0$.
(ii) Let α have minimal polynomial f and let $K = \mathbb{Q}(\alpha)$. Show that the matrix for α with respect to the basis $1, \alpha, \ldots, \alpha^{n-1}$ is C_f and deduce that $\chi_{K,\alpha} = f$.
(iii) Finally let L be a number field and $\alpha \in L$. Show that $\chi_{L,\alpha}(\alpha) = 0$.

8. Let α be an algebraic number with minimal polynomial μ of degree n. We know that
$$\mathbb{Q}(\alpha) = \{a_0 + a_1 + \cdots + a_{n-1}\alpha^{n-1} \ : \ a_i \in \mathbb{Q}\}.$$
Thus every element of $\mathbb{Q}(\alpha)$ can be written (in fact uniquely) as $f(\alpha)$ where $f \in \mathbb{Q}[X]$ has degree $\leq n-1$. It is obvious how to add and subtract elements of $\mathbb{Q}(\alpha)$ when expressed in this form.
(i) Let f, $g \in \mathbb{Q}[X]$ have degree $\leq n-1$. Then $f(\alpha)g(\alpha) = h(\alpha)$ for some $h \in \mathbb{Q}[X]$ with degree $\leq n-1$. Explain how the Euclidean algorithm allows you to compute h.
(ii) Let $f \in \mathbb{Q}[X]$ have degree $\leq n-1$ and suppose $f(\alpha) \neq 0$. Then $f(\alpha)^{-1} = h(\alpha)$ for some $h \in \mathbb{Q}[X]$ with degree $\leq n-1$. Explain how the Euclidean algorithm allows you to compute h.
(iii) Let $\mu = X^3 - 2X - 2$. You showed this to be irreducible in Exercise 6. Let θ be a root of μ. Compute $1/(\theta^2 + 1)$.

9. A field K is **algebraically closed** if every β that is algebraic over K belongs to K.
(i) Explain why $\mathbb{C}$ is algebraically closed.
(ii) Show that $\overline{\mathbb{Q}}$ is algebraically closed.

10. Let $d \in \mathbb{Q}$ be a non-cube and let $\zeta = \exp(2\pi i/3)$. Show that the map
$$\tau \ : \ \mathbb{Q}(\sqrt[3]{d}) \to \mathbb{Q}(\zeta\sqrt[3]{d})$$
given by
$$\tau(a + b\sqrt[3]{d} + c\sqrt[3]{d}^2) = a + b\zeta\sqrt[3]{d} + c\zeta^2\sqrt[3]{d}^2$$
is an isomorphism of fields.
Hint: If you try to do this by simply using the definition of an isomorphism then you just make a mess. Instead look at your lecture notes and find a lemma that gives you this.

11. (i) Let d be a non-square and $K = \mathbb{Q}(\sqrt{d})$. Describe the embedddings σ_1, $\sigma_2 : K \hookrightarrow \mathbb{C}$. What is the signature of K? Are $\sigma_1(K)$ and $\sigma_2(K)$ different?
(ii) Let d be a non-cube and $K = \mathbb{Q}(\sqrt[3]{d})$. Describe the embeddings σ_1, σ_2, $\sigma_3 : K \hookrightarrow \mathbb{C}$. What is the signature of K? Are the $\sigma_j(K)$ different?