Shimura Varieties: Lecture 2 errata

Errata for Lecture 2

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Moduli interpretation for $Y_1(N)$

I got into trouble while attempting to establish the bijection

$$\left\{\begin{matrix}\text{isomorphism classes of pairs } (E, P) \\ \text{where } E \text{ is an elliptic curve over } \mathbb{C} \\ \text{and } P \text{ is a torsion point on } E \text{ of order } N\end{matrix}\right\} \leftrightarrow \Gamma_1(N)\backslash\mathcal{H}$$

The correct argument is as follows:

Suppose that $E = \mathbb{C}/\Lambda$ and $P = x + \Lambda$.

Choose a basis for Λ of the form $\{Nx, y\}$.

Multiplying by $(Nx)^{-1}$, we may assume that our basis is $\{1, \tau\}$ and that $P = \frac{1}{N} + \Lambda$.

An element $\tau' \in \mathcal{H}$ corresponds to the same elliptic curve P precisely when it comes from some other basis for $(c\tau + d, a\tau + b)$ for Λ.

(This was where I went wrong in the lecture: you have to label the basis with $c\tau + d$ as the first element to end up with the usual formula

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

Alternatively, you can follow the common convention of labelling the initial basis as $\{\tau, 1\}$.)

To get the same torsion point P, we must also have

$$\frac{1}{N} + \Lambda = \frac{1}{N}(c\tau + d) + \Lambda$$

which is equivalent to $c \equiv 0$, $d \equiv 1 \mod N$.

Then $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = 1$ implies that $a \equiv 1 \mod N$ as well.

So we end up with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

Line bundles and divisors

Let X be a smooth affine algebraic curve over $\mathbb{C}$. In the lecture I asserted something like: There is a natural bijection

$$\left\{\begin{matrix}\text{isomorphism classes of line bundles} \\ \text{on the smooth compactification of } X\end{matrix}\right\} \leftrightarrow \text{divisors on } X \big/ \text{principal divisors on } X$$

The words "the smooth compactification of" should be removed from the LHS (or wherever I put them in the lecture). Also the condition "affine" is not required.

The following is the correct statement:

For any smooth algebraic curve X over $\mathbb{C}$, there is a natural bijection

$$\left\{\begin{matrix}\text{isomorphism classes of} \\ \text{line bundles on } X\end{matrix}\right\} \leftrightarrow \text{divisors on } X \big/ \text{principal divisors on } X$$