Combinatorial Aspects of Partial Algebras

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Abstract

We extend various parts of the combinatorial theory of semigroups, to encompass the closely related partial algebras of semigroupoids and small categories.

We consider natural definitions of generators and relations for semigroupoids (and hence for small categories). We associate to every semigroupoid a certain categorical-at-zero semigroup, and consider the relationship between presentations for the semigroupoid and the associated semigroup. This allows us to extend to semigroupoids a number of important results concerning the properties of finite generation and finite presentability in semigroups. We also consider the preservation of these properties under Rees matrix constructions over semigroupoids.

We introduce a definition of automaticity for semigroupoids and small categories, which generalises those introduced for semigroups by Hudson and for groupoids by Epstein. We also introduce a definition of prefix-automaticity for semigroupoids and small categories, generalising that for certain monoids introduced by Silva and Steinberg. We consider the relationship between automaticity and prefix-automaticity in a semigroupoid and the associated categorical-at-zero semigroup. This allows us immediately to extend to semigroupoids and small categories a number of results about automatic and prefix-automatic semigroups and monoids. We proceed to consider preservation of automaticity and prefix-automaticity under Rees matrix constructions over semigroupoids, obtaining some results which are new or improve upon existing results even in the semigroup (single-object) case.

In the course of our study, we also prove some new results about automaticity and prefix-automaticity in semigroups and monoids. These include the fact that prefix-automaticity is preserved under the taking of cofinite subsemigroups.

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CHAPTER 1

Introduction

The history of combinatorial algebra is, for the most part, that of combinatorial group theory. Motivated by topological applications, the key ideas of *generators* and *relations* were introduced by Dyck [**24, 25**], and developed into a coherent theory by such luminaries as Poincaré, Dehn, Schreier and Reidemeister. Detailed introductions to combinatorial group theory include [**48, 67**], while a history of the subject can be found in [**16**].

The topological connection has always remained a key element of the subject, and indeed much of the richness of combinatorial group theory comes from its deep connections with geometric group theory. From Reidemeister's celebrated trivialisation of the Nielsen-Schreier Theorem [**59**] right up to the recent theory of automatic groups [**7, 26**], a succession of otherwise opaque results have been rendered transparent through the study of Cayley graphs and presentation 2-complexes as metrizable topological spaces.

In recent years, interest in combinatorial group theory has been given new impetus by computational considerations. Generators and relations form a natural way to describe infinite groups in a computational setting, and much recent research has centred upon issues of decidability and complexity of various problems concerning groups. However, the relationship between groups and computer science is by no means straighforward, and questions involving computational properties of groups have proved difficult to answer.

This is no accident. Intuitively, both the inherent beauty of groups and their ubiquity in nature can be attributed to something *global* about them. Groups are most naturally viewed as a coherent whole and, as tools, are at their most powerful when modelling global properties of things (as in homology and cohomology theory). Computers, by contrast, operate at a *local* level, processing data in a piecewise, sequential manner and discarding irretrievably (uninvertibly!) information which is redundant or has served its purpose. This contrast makes groups rather awkward creatures, from a computational perspective.

For closely related reasons, groups do not readily arise in the study of computation. Rather, computer science tends to give rise to *semigroups* and *monoids*. For example, the study of formal languages is the just the study of subsets of finitely generated free monoids, while the languages recognised by finite automata are the homomorphic pre-images of subsets of finite monoids.

These weaker algebras have been viewed by many pure mathematicians as the poor relations of groups – and not without reason. With less restrictive axioms, they are much looser federations of elements. Deprived of the global properties enjoyed by groups, they seem also to lack much of the intrinsic beauty. However, while the theory of computer science has much of the "feel" of pure mathematics, we must remember that it is really, by definition, applied mathematics. We seek understanding as well as beauty, and if the "ugly" nature of computation compels us to study "ugly" mathematics, then so we must!

These factors, then, lend a new importance to the development of a combinatorial theory of algebras other than groups. In the last decade, significant advances have been made in the combinatorial (as distinct from geometric) theory of semigroups and monoids. A number of authors have considered generators and relations for semigroups, and in particular whether the key properties of finite generation and presentability are preserved under various constructions. A survey of recent work in this area can be found in [**1**], while more details appear in [**2, 3, 4, 6, 10, 11, 12, 41, 60**].

Attempts to marry geometry to the combinatorial study of semigroups have not, in general, met with the same success. Essentially, the absence of inverses in a semigroup lends an inherent notion of *direction* to the associated Cayley graphs, which topology seems ill-equipped to model. Exceptions, where found, tend to involve classes of semigroups which are in some way "group-like" and admit of some local notion akin to that of an inverse in groups. Prime examples include the classes of *inverse* semigroups and *completely simple* semigroups. A truly geometric approach to general semigroups would seem to require some weakened notion of a topology, associated with an "asymmetric metric". No such theory has yet been forthcoming.

Closely related to semigroups and monoids, are the partial algebras known as *semigroupoids* and *small categories*. It has long been known that the basic notions of generators and relations can be generalised to categories [**32, 49**], with graphs and paths substituting for alphabets and words respectively. The same approach applies naturally to semigroupoids. Here, we shall explore the consequences of such a generalisation, showing that some of the recent theory of semigroup presentations generalises to the corresponding notions in semigroupoids (and small categories).

One of the most productive areas of combinatorial (and geometric) group theory in recent years, has been the theory of *automatic groups*, introduced by Epstein *et al* [**26**]. The usual language-theoretic definition of automaticity in groups is rather general, lending itself naturally to application in wider contexts. In [**26**], it is observed that the notion of automaticity, together with many of the associated results, extends naturally from groups to *groupoids*. More recently, Hudson [**43**] introduced a definition of automaticity in semigroups and monoids and an embryonic theory of automatic semigroups has emerged [**13, 14, 15, 21, 23, 33, 34, 36, 37, 38**]. It seems natural to ask whether there is a common framework for these generalisations, that is, a theory of automaticity for small categories and semigroupoids.

We introduce a natural definition of automaticity for semigroupoids (and hence for small categories), and explore the consequences. We consider the relationship between automaticity in semigroupoids and small categories in general, and semigroups and monoids in particular.

Our results have implications for the combinatorial theory not only of semigroupoids in general, but also of semigroups in particular. Some of our results about semigroupoids and small categories are new, even when specialised to the single-object (semigroup or monoid) case. We also prove directly some new results about semigroups, in the course of our study.

In addition, certain classes of semigroups are naturally described by means of constructions over semigroupoids and small categories. Perhaps best known are certain semigroups which are *categorical at zero* [**18**, Section 7.7]. More recently, Lawson [**47**] has introduced a method of finding abundant semigroups, which employs a form of Rees matrix construction over semigroupoids. We show that, under certain assumptions, combinatorial properties such as finite generation, finite presentability and automaticity pass through these constructions.

In addition to this introduction, this thesis comprises eight chapters. In Chapter 2, we introduce the definitions and notation which are necessary in order to understand the sequel.

In Chapter 3, we introduce the key notions of *generators* and *relations* for semigroupoids. We show that a semigroupoid is finitely generated or finitely presentable exactly if a certain associated categorical-at-zero semigroup is finitely generated or finitely presentable, respectively. This allows us to extend to semigroupoids a number of important results concerning presentations for semigroups. In particular, we show that finite generation and presentability are preserved under finite extension of semigroupoids and the taking of cofinite subsemigroupoids.

Chapter 4 considers Rees matrix constructions over semigroupoids. The main results extend those of [**6**], to give characterisations of finite generation and finite presentability in Rees matrix semigroups over semigroupoids, in terms of the corresponding properties in the semigroupoids.

The later chapters focus on notions of *automaticity* and *prefix-automaticity*, as applied to semigroups and to partial algebras. In Chapter 5, we consider the formal language theory and automata theory which of necessity underlies our theory of automaticity. We introduce a notion of a language over a graph, which generalises the usual notion of a language over an alphabet. We define what it means for such a language to be regular. We prove a number of technical lemmas about path languages, some of which may be of independent interest.

Chapter 6 gives a concise but comprehensive survey of the existing theory of automatic and prefix-automatic semigroups and monoids.

In Chapter 7, we prove some miscellaneous new results about automaticity and prefix-automaticity in semigroups, and state some more which will arise as corollaries of later results about semigroupoids. These include the fact that the class of prefix-automatic semigroups is closed under the taking of cofinite subsemigroups, and that automaticity and prefix-automaticity coincide in completely simple and completely 0-simple semigroups, as they do in groups.

In Chapter 8, we introduce a definition of automaticity for general finitely generated semigroupoids. In an echo of Chapter 3, we show that a semigroupoid has the property of automaticity or prefix-automaticity if and only if the associated categorical-at-zero semigroup has the same property. This allows us to generalise a considerable body of theory concerning automatic semigroups to automatic semigroupoids, including both existing results and some of our new ones.

Finally, in Chapter 9, we consider the relationship between automaticity and prefix-automaticity in semigroupoids, and Rees matrix constructions over semigroupoids. We show that a finitely generated Rees matrix semigroup over an automatic semigroupoid is always automatic, and provide some sufficient conditions for the underlying semigroupoid of an automatic Rees matrix semigroup to be automatic. We also prove some related results for prefix-automaticity.

CHAPTER 2

Preliminaries

In this chapter, we briefly introduce some definitions and associated notation which are necessary to make this thesis self-contained. A comprehensive survey of classical semigroup theory, combinatorial algebra or the theory of partial algebras would be beyond the scope of this work. The reader in search of a more detailed introduction is directed to [**17, 18, 40**] for semigroup theory, [**48, 67**] for combinatorial group theory and [**32, 49**] for category theory. Semigroupoids are widely discussed in the literature on category theory, although rarely a subject of study in their own right.

We postpone discussion of some more specialised topics, which will not be needed until the later chapters. In particular, *formal languages*, *finite automata* and *regularity* are introduced in Chapter 5, while *automatic* groups, monoids and semigroups are covered in Chapter 6.

Some of the material in this chapter appeared in the preprint [**45**].

2.1. Basic Notation

Throughout this thesis, function applications are written on the right and application brackets omitted where no ambiguity results. Thus, we write $x\sigma$ instead of $\sigma(x)$, for the result of applying a function σ to an element x of its domain. Accordingly, function composition is denoted by concatenation and read from left to right. This arrangement, now frequently adopted in the literature of combinatorial algebra, is more intuitive for speakers of European languages, and accords with the convention that automata read words from left to right.

We denote by $\mathbb{N}$ the set of non-negative integers *including zero*, that is, the set of cardinalities of finite sets.

For a positive integer j we will always denote by π_j the natural projection which takes an n-tuple ($n \geq j$) to its jth coordinate.

2.2. Semigroups, Monoids and Groups

A *semigroup* is a non-empty set with an associative binary operation defined on it, usually written as juxtaposition. For brevity, it is usual to identify a semigroup with its set of elements, and take the operation as read. A *monoid* M is a semigroup with an *identity*, that is, an element $e \in M$ with the property that $ex = x = xe$ for all $x \in M$. A *group* G is a monoid with the property that, for every element $x \in G$, there is an element $x^{-1} \in G$ (called the *inverse* of x) such that xx^{-1} is the identity in G.

The multiplication in a semigroup S induces a multiplication on the power set of S, given by

$$AB = \{ab \mid a \in A, b \in B\} \subseteq S$$

for all $A, B \subseteq S$.

A *subsemigroup* [*subgroup*] of a semigroup S is a subset of S which forms a semigroup [respectively, a group] under the same operation. A *submonoid* of a monoid S is a subsemigroup (necessarily a monoid) of S containing the identity element of S. Note that a subsemigroup of a monoid which is itself a monoid need not be a submonoid, since it may have a different identity element. A subsemigroup I of a semigroup S is called a *left ideal* [*right ideal*] if $SI \subseteq I$ [respectively, $IS \subseteq I$]. A subsemigroup which is both a left ideal and a right ideal is called an *ideal*.

A subsemigroup, submonoid or ideal of S is called *proper* if it is a proper subset of S, that is, if it does not contain all the elements of S.

Let S be a semigroup. An element $0 \in S$ is a called a *zero* in S if $x0 = 0 = 0x$ for all $x \in S$. Clearly, a semigroup can have at most one zero.

Let I be an ideal of a semigroup S. Let 0 be a new symbol, not in S. The *Rees quotient* of S by I is the semigroup with set of elements $S \setminus I \cup \{0\}$ and multiplication given by

$$st = \begin{cases} \text{the } S\text{-product } st & \text{if } s, t \in S \text{ and the } S\text{-product } st \text{ is in } S \setminus I \\ 0 & \text{otherwise} \end{cases}$$

for all $s, t \in S \setminus I \cup \{0\}$.

A semigroup S is *finitely generated* if there is a finite subset $T \subseteq S$ such that every element of S can be written as a product of elements of T. Following [**65**], we say that a finitely generated semigroup S has *finite geometric type* if for every $p \in S$ there exists a constant $k \in \mathbb{N}$ such that for every $q \in S$, the set $\{x \in S \mid xp = q\}$ has cardinality at most k. Geometrically, this constitutes a requirement that every Cayley graph of S with respect to a finite generating set has an upper bound on the degree of its vertices. This property is also discussed in [**33, 37**] in the case of commutative semigroups, where it is called *bounded indegree*.

2.3. Completely Simple and Completely 0-Simple Semigroups

Let S be a semigroup without zero. An element $e \in S$ is said to be *idempotent* if $ee = e$. We define a partial order on the idempotents of S, by $e \leq f$ if and only if $ef = fe = e$. An idempotent is called *primitive* if it is minimal amongst idempotents with respect to this partial order. We say that S is *simple* if it has no proper ideals. We say that S is *completely simple* if it is simple and has a primitive idempotent.

Suppose now that S is a semigroup with a zero 0. We say now that an idempotent e is *primitive* if it is minimal amongst non-zero idempotents with respect to the order $\leq$ defined above. Clearly, $\{0\}$ is an ideal of S. We say that S is 0-*simple* if its only ideals are S and $\{0\}$, and also

$$S^2 = \{st \mid s, t \in S\} \neq \{0\}.$$

This latter technical requirement serves to exclude the two element null semigroup. We say that S is *completely* 0*-simple* if it is 0-simple and has a primitive idempotent.

Let G be a group, and let Λ and I be non-empty sets. Let P be a $\Lambda \times I$ matrix with entries drawn from G. We define a semigroup $M(G; I, \Lambda; P)$ with set of elements $I \times G \times \Lambda$ and multiplication given by

$$(i, g, \lambda)(j, h, \mu) = (i, gP_{\lambda j}h, \mu).$$

where $P_{\lambda j}$ denotes the entry of P in row λ and column j. A semigroup which arises in this way, for some G, Λ, I and P, is called a *Rees matrix semigroup* over the group G. Suschkewitz [**68**] proved a result equivalent to the fact that a semigroup is completely simple exactly if it is isomorphic to a Rees matrix semigroup over a group.

A similar, if slightly more complex, construction characterises the completely 0-simple semigroups. Let G be a group, and let G^0 denote the semigroup (or 0*-group*) obtained by adjoining a zero element 0 to G. Let Λ and I be non-empty sets. Let P be a $\Lambda \times I$ matrix with entries drawn from G^0, such that every row and every column of P contains at least one non-zero entry. Define a semigroup $M^0(G; I, \Lambda; P)$ with set of elements $(I \times G \times \Lambda) \cup \{0\}$ and multiplication given by

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gP_{\lambda j}h, \mu) & \text{if } P_{\lambda j} \in G \\ 0 & \text{if } P_{\lambda j} = 0 \end{cases}$$

and $0x = x0 = 0$ for all $x \in (I \times G \times \Lambda) \cup \{0\}$.

A semigroup which arises in this way, for some G, Λ, I and P, is called *Rees matrix semigroup with zero* over the group G. Rees [**58**] showed that a semigroup is completely 0-simple exactly if it is isomorphic to a Rees matrix semigroup with zero over a group, with a sandwich matrix in which every row and column contains a non-zero entry. See [**40**, Chapter 3] for a detailed exposition of both of the above results. More general Rees matrix constructions will be introduced in Chapter 4, and will also feature in Chapters 7 and 9.

2.4. Graphs

For our purposes, a *(directed) graph* X consists of a collection X^0 of *vertices*, a collection X^1 of *edges* and two functions $\omega, \alpha : X^1 \to X^0$ which determine respectively the *source* and *target* of each edge.

Note that this definition allows multiple edges joining the same pair of vertices, and edges which start and end at the same vertex. We say that X is *small* if X^0 and X^1 are sets, and *finite* if X^0 and X^1 are finite sets.

A *path* π in X is a finite sequence $e_1 e_2 \dots e_n$ of (not necessarily distinct) edges in X such that $e_i\alpha = e_{i+1}\omega$ for $1 \leq i < n$. We define $\pi\omega = e_1\omega$, $\pi\alpha = e_n\alpha$, and call these the *source* and *target* respectively of the path π. The *length* of the path π is the number n of edges. We also define a distinct *empty path* of length 0 at each vertex, with source and target that vertex. We identify each path of length 1 with its single edge, and each vertex with the empty path at that vertex. Thus, X^0 and X^1 are the sets of paths in X of length 0 and of length 1 respectively. We shall further exploit this notation in Section 5.2 below.

Given (finite) graphs X and Y, the *direct product* of X and Y is the (finite) graph with vertex set $X^0 \times Y^0$, and edge set $X^1 \times Y^1$ with $(f, g)\alpha = (f\alpha, g\alpha)$ and $(f, g)\omega = (f\omega, g\omega)$ for all $f \in X^1, g \in Y^1$. We denote this graph $X \times Y$.

Let X and Y be graphs. For our purposes, a *graph morphism* $\sigma : X \to Y$ consists of functions $\sigma^0 : X^0 \to Y^0$ and $\sigma^1 : X^1 \to Y^1$ such that $x\sigma^1\alpha = x\alpha\sigma^0$ and $x\sigma^1\omega = x\omega\sigma^0$ for all edges $x \in X^1$.

If, in addition, $X^0 = Y^0$ and $\sigma^0 : X^0 \to X^0$ is the identity function, then we call σ an *edge-morphism*, and, for notational convenience, identify σ with σ^1.

2.5. Semigroupoids and Small Categories

A *semigroupoid* S is a small graph S, together with a partial multiplication on the edges of S such that, for any two edges $e, f \in S^1$

(i) the product ef is defined if and only if $e\alpha = f\omega$;

(ii) if ef is defined, then $(ef)\omega = e\omega$ and $(ef)\alpha = f\alpha$; and

(iii) if ef and fg are defined then $e(fg) = (ef)g$.

The vertices and edges of a semigroupoid are called respectively *objects* and *arrows*. Where no ambiguity arises, we abuse the notation slightly by writing S to mean the set S^1 of arrows in S.

If $s_0 \in S^0$ is such that there exists an $s \in S^1$ with $s\alpha = s\omega = s_0$, we say that the *local semigroup* of S at the object s_0 is the semigroup with elements $\{s \in S \mid s\alpha = s\omega = s_0\}$ and multiplication defined by restricting that in S. A semigroupoid is *strongly connected* if for every pair $s_0, t_0 \in S^0$ of objects, there is an arrow (or equivalently, a non-empty path) in S with source s_0 and target t_0.

An object $x \in S^0$ of a semigroupoid S is called *isolated* if is not at the start or end of any arrow, that is, if $x \notin S^1\alpha \cup S^1\omega$. A semigroupoid is called *isolation-free* if it has no isolated objects.

The partial multiplication in a semigroupoid S induces a natural multiplication on the subsets of S, given by

$$AB = \{ab \mid a \in A, b \in B, a\alpha = b\omega\}$$

for all $A, B \subseteq S$.

Let S and T be semigroupoids with the same set of objects. A *(semigroupoid) morphism* is a function $\sigma : S^1 \to T^1$ such that $\sigma\alpha = \alpha$, $\sigma\omega = \omega$ and $(s\sigma)(t\sigma) = (st)\sigma$ for all $s, t \in S$ with $s\alpha = t\omega$. Thus, a morphism of semigroupoids is an edge-morphism of the underlying graphs, which preserves the multiplication where defined. As an example of the abuse of notation discussed above, we shall sometimes identify $\sigma : S \to T$ with $\sigma^1 : S^1 \to T^1$.

An arrow $e \in S$ is called a *(local) identity* at the object $e\alpha \in S^0$ if $e\alpha = e\omega$, and $ef = f$ for all arrows f such that ef is defined, and $fe = f$ for all arrows f such that fe is defined. There is, necessarily, at most one local identity at a given object. We write 1_s to denote the unique local identity at the object $s \in S_0$, if it exists. A semigroupoid with an identity at every object is called a *small category*. For a semigroupoid to be a small category, it is necessary (but not sufficient) that each of its local semigroups be a monoid. An arrow e is said to be an *inverse* of an arrow f if ef and fe are both defined and equal the identities at $f\alpha$ and $e\alpha$ respectively. A small category in which every arrow has an inverse is called a *groupoid*.

A *subsemigroupoid* of a semigroupoid S is a semigroupoid U where $U^0 \subseteq S^0$, $U^1 \subseteq S^1$ and the multiplication, source function ω and target function α on U are the restrictions of those on S to the appropriate domains. A *subcategory* of a category S is a subsemigroupoid U such that for every $u \in U^0$, the local identity $1_u \in S^1$ is contained in U^1. We say that U is a *cofinite* subsemigroupoid (or *cofinite* subcategory) of S, if $S^1 \setminus U^1$ is finite. If A is a subset of a subsemigroupoid [subcategory] S then the subsemigroupoid [subcategory] *generated* by A is the unique minimum (with respect to inclusion) subsemigroupoid [subcategory] of S which contains A.

CHAPTER 3

A Combinatorial Approach to Partial Algebras

It has long been known that the basic combinatorial concepts of *generators* and *relations* generalise from groups and semigroups to groupoids and small categories. In this chapter, we show that it is straightforward further to generalise these ideas to semigroupoids.

We then associate to every semigroupoid a certain semigroup, and show that many of the combinatorial properties of the semigroupoid are shared by the associated semigroup. This allows us to extend to semigroupoids some of the existing theory of presentations for semigroups.

Most of the material in this chapter previously appeared in the author's preprint [**45**]. No particular claim of originality is made in respect of the ideas in Section 3.1, most of which appear to be common knowledge, at least in the case of small categories. However, the author has been unable to discover an explicit exposition in the literature, and hence feels the need to provide one here.

3.1. Free Semigroupoids and Presentations

Let X be a small graph. The *free semigroupoid* X^+ on X is the semigroupoid with

- objects the vertices of X;
- arrows the *non-empty* paths in X;
- source and target functions defined as for paths; and
- partial multiplication defined by concatenation of paths.

The *free category* X^* is the category obtained by adjoining a local identity (which can be thought of as the empty path) at each object of X^+. The terminology is justified by the following proposition.

PROPOSITION 3.1. *Let X be a graph, S a semigroupoid [respectively, a small category] with $S^0 = X^0$, and $\phi : X \to S$ an edge-morphism. Then there is a unique extension of ϕ to a semigroupoid morphism from X^+ [respectively, X^*] to S.*

PROOF. Clearly, we can well-define a semigroupoid morphism
$\phi' : X^+ \to S$ by

$$(e_1 e_2 \dots e_n)\phi' = (e_1\phi)(e_2\phi)\dots(e_n\phi)$$

for all paths $e_1e_2\dots e_n \in X^+$. Furthermore, for any semigroupoid morphism $\theta : X^+ \to S$ extending ϕ, we must have

$$\begin{aligned}(e_1e_2\dots e_n)\theta &= (e_1\theta)(e_2\theta)\dots(e_n\theta)\\ &= (e_1\phi)(e_2\phi)\dots(e_n\phi)\\ &= (e_1e_2\dots e_n)\phi'\end{aligned}$$

for all $e_1e_2\dots e_n \in X^+$, so that $\theta = \phi'$.

A similar argument applies in the small category case. $\square$

Let S be a semigroupoid. A *congruence* on S is a binary equivalence relation ρ on the arrows of S such that

(i) if $e\rho f$ then $e\alpha = f\alpha$ and $e\omega = f\omega$; and
(ii) if $e\rho f$, $g\rho h$ and eg and fh are defined then $(eg)\rho(fh)$.

We denote by S/ρ the semigroupoid with

- object set S^0;
- arrows the equivalence classes of ρ;
- $[s]\omega = s\omega$ for all $s \in S^1$;
- $[s]\alpha = s\alpha$ for all $s \in S^1$; and

- $[s][t] = [st]$ for all $s, t \in S$ with $s\alpha = t\omega$.

The conditions (i) and (ii) serve to ensure that the start and terminal vertices and the multiplication in S/ρ are well-defined.

A pair $(u, v) \subseteq X^+ \times X^+$ of paths with the same start and end point is called a *relation*. To avoid confusion with the other common use of this term, we shall be careful to use the term "binary relation" for a subset of a Cartesian product of two sets.

Given a morphism from X^+ onto S, a relation (u, v) is said to be *satisfied* in S if u and v *represent* (map onto) the same element in S. Two paths $x, y \in X^+$ are said to be *related by an application* of the relation (u, v) if either $x = sut$ and $y = svt$ or $x = svt$ and $y = sut$ for some $s, t \in X^*$.

Let $R \subseteq X^+ \times X^+$ be a set of relations. Then x and y are *related as a consequence of* R if there exists an $n \geq 1$ and paths $x = p_1, p_2, p_3, \ldots, p_n = y$ such that for $1 \leq i < n$, p_i is related to p_{i+1} by an application of some relation in R. We define

$$R^\# = \{(x, y) \in X^+ \times X^+ \mid x \text{ and } y \text{ are related as a consequence of } R\}.$$

The following proposition provides an alternative characterisation of $R^\#$.

PROPOSITION 3.2. *The binary relation $R^\#$ is the unique minimum (with respect to set inclusion) congruence on X^+ containing R.*

PROOF. Reflexitivity, symmetry and transitivity of $R^\#$ follow directly from the definitions, as does the fact that $R^\#$ satisfies properties (i) and (ii) above.

To see that $R^\#$ is minimum amongst congruences containing R, suppose R' is also a congruence on X^+ which contains R. Let $(u, v) \in R^\#$. Then u and v are related as a consequence of R, so there exists $n \geq 1$ and paths $u = p_1, p_2, p_3, \ldots, p_n = v$ such that for $1 \leq i < n$, the path p_i is related to p_{i+1} by an application of some relation in R. This means that for $1 \leq i < n$, we have $p_i = s_i u_i t_i$ and $p_{i+1} = s_i v_i t_i$ for some relation $(u_i, v_i) \in R \subseteq R'$. Since R'

is a congruence, it follows that each $(p_i, p_{i+1}) \in R'$. Now by transitivity of R', it follows that $(u, v) \in R'$. Thus $R^{\#} \subseteq R'$, justifying the claim that $R^{\#}$ is the minimum congruence containing R. $\square$

Let S be a semigroupoid, X be a small graph, and R be a set of relations on the free semigroupoid X^+, that is, a binary relation which satisfies property (i) above. We say that $\langle X \mid R \rangle$ is a *presentation* for S, if S is isomorphic to $X^+/R^{\#}$. It is straightforward to show that every semigroupoid has a presentation. A presentation $\langle X \mid R \rangle$ is called *finitely generated* if X^1 is finite, that is, if X has finitely many edges. A presentation $\langle X \mid R \rangle$ is called *finite* if X^1 and R are both finite. A semigroupoid with a finitely generated presentation is called *finitely generated* while a semigroupoid with a finite presentation is called *finitely presentable*.

It is readily verified that a semigroupoid S is finitely generated exactly if there exists a finite subset $Y \subseteq S^1$ such that every arrow in S^1 can be written as a product of arrows in Y. Thus, the definition given here is consistent with that of a finitely generated semigroup introduced in Section 2.2.

We emphasise that, in order for a semigroupoid presentation $\langle X \mid R \rangle$ to be finite or finitely generated, we do *not* require the graph X to have finitely many *vertices*. This approach simplifies the treatment later, and is of relevance only when considering semigroupoids which have infinitely many isolated vertices. In particular, if the semigroupoid is in fact a small category, it would be equivalent to require that the graph X be finite.

The following alternative characterisation of a presentation is a simple consequence of the definition.

PROPOSITION 3.3. *Let X be a small graph, and R a set of relations on X. Then $\langle X \mid R \rangle$ is a presentation for S if and only if there exists a surjective morphism $\sigma : X^+ \to S$ with respect to which all relations in R are satisfied in S, and all relations satisfied in S occur as consequences of relations in R.*

To relate all the above definitions to the usual concepts in combinatorial algebra, we observe that isolation-free, single-object semigroupoids [respectively, small categories] are in a natural bijective correspondence with their respective (unique) local semigroups [respectively, monoids]. Extending this correspondence to presentations, we see that single-vertex small graphs correspond to *alphabets* and paths in such graphs to *words*. It is straightforward to verify that, in the single-object case, our other definitions simplify to coincide with those usual in the theory of semigroups and monoids.

3.2. Semigroupoids and Categorical-at-zero Semigroups

Let T be a semigroup with a zero 0. We say that T is *categorical at zero*, if for any elements $a, b, c \in T$ such that $ab \neq 0$ and $bc \neq 0$, we also have that $abc \neq 0$. [**18**, Section 7.7]

Now let S be a semigroupoid and 0 be a new symbol not in S^1. Let $T = S^1 \cup \{0\}$, and define a multiplication on T by

$$st = \begin{cases} \text{the } S\text{-product } st & \text{if } \ s, t \in S \text{ and } s\alpha = t\omega \\ 0 & \text{otherwise} \end{cases}$$

for all $s, t \in S$. Clearly, T is a semigroup with zero 0, and T is categorical-at-zero. We shall call T the *(categorical-at-zero) semigroup associated with* S.

Note that not every categorical-at-zero semigroup is associated with a semigroupoid. Indeed, if T is associated with S as above then clearly T has the additional property that if $a, a', b, b' \in T$ are such that $ab \neq 0$, $ab' \neq 0$ and $a'b \neq 0$ then also $a'b' \neq 0$. Consider the semigroup with presentation

$$\langle a, b, a', b', 0 \mid a0 = b0 = a'0 = b'0 = 00 = 0a = 0b \ = 0a' = 0b' = a'b' = 0\rangle.$$

Clearly the generator 0 represents a zero element of the semigroup, which we shall also denote by 0. To see that this semigroup is categorical at zero, let

x, y, z be elements, and suppose for a contradiction that $xy \neq 0$, $yz \neq 0$ and $xyz = 0$. Let x', y' and z' be non-empty words in the generators a, b, a', b' and 0 representing the elements x, y and z respectively. Then the word $x'y'z'$ must be reducible (eventually to 0) by applying the given relations. But the relations can be applied only if $x'y'z'$ contains 0 or $a'b'$ as a factor, in which case either $x'y'$ or $y'z'$ contains a 0 or $a'b'$ as a factor. It follows that either $xy = 0$ or $yz = 0$, giving the required contradiction.

Thus, this semigroup is categorical at zero. However, $a'b'$ represents the zero element, but none of the relations can be applied to any of ab, ab' or $a'b$, so they must represent non-zero elements. Thus, the categorical-at-zero semigroup with the given presentation is not associated to any semigroupoid.

The following theorem will allow us to generalise a number of known results about semigroup presentations to the wider case of semigroupoid presentations.

THEOREM 3.4. *Let T be the categorical-at-zero semigroup associated with a semigroupoid S. Then S is a finitely generated [finitely presentable] semigroupoid if and only if T is a finitely generated [finitely presentable] semigroup.*

PROOF. First, suppose S is finitely generated by a subset $Y \subseteq S$. Then it is easy to see that the finite set $Y \cup \{0\}$ generates T. Conversely, if T is finitely generated by a subset $Z \subseteq T$, then the finite set $Z \cap S$ generates S.

Now suppose S has a finite presentation $\langle X \mid R \rangle$. Let A be the (finite) set X^1 of edges of X, and let $\phi : X^+ \to A^+$ be defined in the obvious way. Let z be a symbol not in X which will represent the zero element in T. Let

$$
\begin{aligned}
R' \quad &= \quad \{(v\phi, w\phi) \mid (v, w) \in R\} && (1) \\
&\cup \quad \{(za, z), (az, z) \mid a \in A \cup \{z\}\} && (2) \\
&\cup \quad \{(ab, z) \mid a, b \in A, a\alpha \neq b\omega\} && (3)
\end{aligned}
$$

We claim that $\langle A \cup \{z\} \mid R' \rangle$ is a finite presentation for T with the obvious isomorphism. It is straightforward to verify that R' is finite and that all the relations in R' hold in T, and it follows from our argument above that T is generated by the image of $A \cup \{z\}$. It remains to show that all relations which hold in T are consequences of the relations in R'.

Suppose $a = a_1 a_2 \dots a_m$ and $b = b_1 b_2 \dots b_n$ are words in $(A \cup \{z\})^+$ such that the relation (a, b) holds in T. Let $t \in T$ be the element represented by a (and by b). First, consider the case where $t = 0$. If a contains the symbol z, then a is related to z by a chain of applications of relations from (2). Otherwise, $a \in A^+$ but clearly a does not represent a path in X. It follows that there must exist i with $1 \leq i < m$ such that $a_i \alpha \neq a_{i+1} \omega$. Now a is related by an application of a relation from (3) to $a' = a_1 \dots a_{i-1} z a_{i+2} \dots a_m$, which in turn is related by a chain of applications of relations from (2) to z. A similar argument shows that b is related by a chain of applications of relations in R' to z, and thence to a.

Otherwise, if $t \neq 0$ then a and b correspond to paths in X, which represent the element t as an arrow of S. It follows easily that a and b are represented by a chain of applications of relations from (1), as required.

Conversely, suppose $\langle A \mid R \rangle$ is a finite presentation for T. Let B be the set of letters in A which do not represent zero. For each $b \in B$, define $b\omega$ and $b\alpha$ to be respectively the source and target of the element of T represented by b, when considered as an arrow in S. Let X be the graph with vertex set S^0, edge set B and source and target functions ω and α as just defined. Let $\phi : X^+ \to A^+$ be defined in the obvious way.

We observe that the image $X^+\phi$ of ϕ is exactly the set of words in A^+ which do not represent 0. Clearly, since $\{0\}$ is an ideal of T, $X^+\phi$ is closed under the taking of factors. Also, if $X^+\phi$ contains a representative for an element $x \in T$, then it contains all representatives in A^+ for x. It follows that the only relations in R which can be applied to words in $X^+\phi$ are those in

$R \cap (X^+\phi \times X^+\phi)$. With this in mind, we define a set of relations

$$R' = \{(v, w) \mid (v\phi, w\phi) \in R\}.$$

It is readily verified that the function ϕ is injective, and since R is finite, it follows that R' is finite. We claim that $\langle X \mid R' \rangle$ is a finite presentation for S. Let $\sigma : X^+ \to S$ be the obvious morphism.

It is easy to verify that S is generated by the image of X under σ, and that relations in R' are satisfied in S. Now suppose a relation (v, w) holds in S. Then certainly also the relation $(v\phi, w\phi)$ holds in T, so $v\phi$ and $w\phi$ are related by a chain of applications of relations in R. But by our observations above, these relations must all be contained in $X^+\phi \times X^+\phi$, and so have counterparts in R'. It follows easily that v and w are related by a chain of applications of relations in R'. □

3.3. Some Properties of Semigroupoid Presentations

The main result of the previous section allows us immediately to extend to semigroupoids an important result about finite generation and finite presentability of semigroups.

THEOREM 3.5. *Let U be a cofinite subsemigroupoid of a semigroupoid S. Then S is finitely generated [finitely presentable] if and only if U is finitely generated [respectively, finitely presentable].*

PROOF. Let T and V be the categorical-at-zero semigroups associated with S and U respectively. Clearly, V is a cofinite subsemigroup of T. Now by Theorem 3.4, S is finitely generated [finitely presentable] if and only if T is finitely generated [finitely presentable]. By Theorem 1.1 [Theorem 1.3] of [**62**], T is finitely generated [finitely presentable] if and only if V is finitely generated [finitely presentable]. Finally, by Theorem 3.4, V is finitely generated [finitely presentable] if and only if U is finitely generated [finitely presentable]. □

The following result, the semigroup analogue of which is well known [**18**, Section 9.2], says that a finitely presentable semigroupoid has a finite presentation with respect to every finite choice of generators.

THEOREM 3.6. *Let S be a finitely presentable semigroupoid, and Y be a graph with finitely many edges, such that there exists a natural surjective morphism from Y^+ onto S. Then there exists a finite set of relations $R' \subseteq Y^+ \times Y^+$ such that $\langle Y \mid R' \rangle$ is a (finite) presentation for S.*

PROOF. Suppose S has a finite presentation $\langle X \mid R \rangle$. For each edge $y \in Y$, let $\lambda_y \in X^+$ be a path in X representing the same arrow as y. Similarly, for each edge $x \in X$, let $\rho_x \in Y^+$ be a path in Y representing the same arrow as x. Define a set of relations

$$
\begin{aligned}
R' \quad &= \quad \{(\rho_{x_1}\rho_{x_2}\dots\rho_{x_m}, \rho_{x'_1}\dots\rho_{x'_n}) \mid (x_1\dots x_m, x'_1\dots x'_n) \in R\} &(4)\\
&\cup \quad \{(y, \rho_{x_1}\dots\rho_{x_n}) \mid y \in Y, \lambda_y = x_1\dots x_n\}. &(5)
\end{aligned}
$$

We claim that $\langle Y \mid R' \rangle$ is a finite presentation for S. It is straightforward to check that R' is finite and that relations in R' are satisfied in S.

Suppose now that a relation $(u, u') \in Y^+ \times Y^+$ is satisfied in S. Then u and u' are related by applications of relations in (5) to words of the form $\rho_{x_1}\dots\rho_{x_m}$ and $\rho_{x'_1}\dots\rho_{x'_n}$ respectively. Now the relation

$$(x_1\dots x_m, x'_1\dots x'_n) \in X^+ \times X^+$$

is satisfied in S, and $\langle X \mid R \rangle$ is a presentation for S, so it follows that $x_1\dots x_m$ and $x'_1\dots x'_n$ are related by a chain of applications of relations in R. But it now follows easily that $\rho_{x_1}\dots\rho_{x_m}$ and $\rho_{x'_1}\dots\rho_{x'_n}$ are related by a chain of applications of relations in (4).

It follows that $\langle Y \mid R \rangle$ is a finite presentation for S, as required. $\square$

CHAPTER 4

Rees Matrix Constructions

We saw in Chapter 2 that Rees matrix constructions over groups were introduced by Rees [**58**], and used to provide characterisations of the completely simple and completely 0-simple semigroups. While for that purpose, it is only necessary to consider Rees matrix semigroups over *groups*, the validity of the general construction is not dependent upon the group structure. For example, McAlister [**50**] made use of Rees matrix constructions over inverse semigroups to construct locally inverse regular semigroups.

Rees matrix constructions over *semigroupoids* were introduced by Lawson [**47**], who used them to construct a class of locally adequate abundant semigroups. They represent an alternative formulation of certain special cases of the *blocked Rees matrix semigroup* constructions introduced by Fountain [**27**] and subsequently employed by Armstrong [**5**]. We extend Lawson's definition in a natural way to define Rees matrix semigroups *with zero* over semigroupoids. This slightly more general construct reformulates Fountain's approach in the general case. We then proceed to consider the relationship between the combinatorial properties of a semigroupoid and of Rees matrix semigroups over that semigroupoid, extending some results of [**6**].

Some of the material in this chapter previously appeared in the preprint [**45**].

4.1. Rees Matrix Constructions

Let S be a non-empty, isolation-free semigroupoid, and I and Λ be indexing sets. Let

$$F : I \to S^1\omega \subseteq S^0$$

and

$$G : \Lambda \to S^1\alpha \subseteq S^0$$

be surjective functions. Let 0 be a new symbol not in S, and let P be a $\Lambda \times I$ matrix with entries drawn from $S \cup \{0\}$, with the property that $P_{\lambda i}\alpha = iF$ and $P_{\lambda i}\omega = \lambda G$ for all $i \in I, \lambda \in \Lambda$ such that $P_{\lambda i} \neq 0$. The *Rees matrix semigroup with zero* $M = M^0(S; F(I), G(\Lambda); P)$ is the semigroup with set of elements

$$M = \{0\} \cup \{(i, x, \lambda) \in I \times S \times \Lambda \mid iF = x\omega, x\alpha = \lambda G\}$$

and multiplication given by

$$(i, x, \lambda)(j, y, \mu) = \begin{cases} (i, xP_{\lambda j}y, \mu) & \text{if } P_{\lambda j} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $(i, x, \lambda), (j, y, \mu) \in M \setminus \{0\}$, and

$$0m = m0 = 0$$

for all $m \in M$.

Note that in the expression "$M^0(S; F(I), G(\Lambda); P)$", the use of the notation "$F(I)$" and "$G(\Lambda)$" is purely symbolic. It is intended to remind the reader of the relationship between F and I and between G and Λ.

We call P the *sandwich matrix* of the construction. If P contains no zero entries, then $M \setminus \{0\}$ is a subsemigroup of M, which we call the *Rees matrix semigroup (without zero)* $M(S; F(I), G(\Lambda); P)$.

We remark briefly upon the choice of codomains for the indexing functions F and G. This restriction does not limit the range of semigroups which appear as Rees matrix semigroups. Indeed, if I, Λ, F, G, S and P, do not satisfy this requirement but otherwise satisfy the requirements for the Rees matrix construction, then one can instead perform the construction with subsets of I and Λ, a subsemigroupoid of S and the corresponding restrictions of F and G and a submatrix of P, to obtain the same Rees matrix semigroup as would be obtained by using the construction on the more general semigroupoid. Similarly, given a semigroupoid S with isolated objects, one can remove those objects to obtain an isolation-free semigroupoid before using a Rees matrix construction. The purpose of the restrictions is to allow the following straightforward proposition.

PROPOSITION 4.1. *Let*

$$M = M^0(S; F(I), G(\Lambda); P)$$

or

$$M = M(S; F(I), G(\Lambda); P)$$

be a Rees matrix semigroup with or without zero over a semigroupoid S. Then

- *for every $i \in I$ there exist $s \in S^1$ and $\lambda \in \Lambda$ with $(i, s, \lambda) \in M$;*
- *for every $s \in S^1$ there exist $i \in I$ and $\lambda \in \Lambda$ with $(i, s, \lambda) \in M$;*
- *for every $\lambda \in \Lambda$ there exist $i \in I$ and $s \in S^1$ with $(i, s, \lambda) \in M$; and*
- *for every $v \in S^0$ there exists $i \in I$ with $iF = v$ or $\lambda \in \Lambda$ with $\lambda G = v$.*

4.2. Rees Matrices and Categorical-at-zero Semigroups

Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup over a semigroupoid S, and let T be the categorical-at-zero semigroup associated to S (see Section 3.2). We consider the relationship between M and the Rees matrix semigroup $M' = M^0(T; I, \Lambda; P)$ over T. Note that the zero element in T is distinct from that in M and from the zero entries which may occur in the matrix P. For clarity, in this section, We shall denote these by 0_T, 0_M and 0_P respectively. For convenience, we shall identify the zero element in M' with that in M.

By definition, the semigroup M' has set of elements

$$(I \times (S \cup \{0_T\}) \times \Lambda) \cup \{0_M\}.$$

Let

$$J = \{(i, s, \lambda) \mid s \in S, iF = s\omega, s\alpha = \lambda G\} = M \setminus \{0_M\}.$$

Then J is a subset of M'. We claim that $M' \setminus J$ is an ideal of M', and that the Rees quotient of M' by $M' \setminus J$ is isomorphic to M.

To show that $M' \setminus J$ is a right ideal, let $x \in M' \setminus J$ and $m \in M'$. If $x = 0_M$ or $m = 0_M$ then clearly $mx = 0_M \in M' \setminus J$. Otherwise, we have $x = (i, t, \lambda)$ and $m = (i', t', \lambda')$ for some $t, t' \in T$. If $P_{\lambda i'} = 0_P$ then $xm = 0_M \in M' \setminus J$. Otherwise, we have

$$xm = (i, tP_{\lambda i'}t', \lambda').$$

If $t = 0_T$ or $t' = 0_T$ then we have

$$xm = (i, 0_T, \lambda') \in M' \setminus J$$

and we are done, so assume now that $t, t' \in S$. Since $x \notin J$, we must have either $iF \neq t\omega$ or $t\alpha \neq \lambda G$ or both. If the latter holds, then we have

$$t\alpha \neq \lambda G = P_{\lambda i'}\omega$$

so that $tP_{\lambda i'}t' = 0_T$. If only the former holds, then we have

$$iF \neq t\omega = (tP_{\lambda i'}t')\omega.$$

In both cases, we have $xm \in M' \setminus J$, as required. An entirely similar argument shows that $M' \setminus J$ is a left ideal, and it follows that $M' \setminus J$ is an ideal.

It is now immediate from the definitions that the Rees quotient of M' by $M' \setminus J$ is isomorphic to (indeed, if we identify the respective zero elements, identical to) the semigroup M. Furthermore, it is clear from the definition that the multiplication in M is a restriction of that in M'. It follows that $M = J \cup \{0_M\}$ is a subsemigroup of M', also isomorphic to M.

4.3. Combinatorial Properties of Rees Matrix Semigroups

In the remainder of this chapter, we extend some of the results of [**6**] concerning Rees matrix semigroups over semigroups, to Rees matrix semigroups over semigroupoids. We observed above that a Rees matrix semigroup over a semigroupoid is necessarily a Rees quotient of a Rees matrix semigroup over the associated categorical-at-zero semigroup. We saw in the previous chapter that a semigroupoid is finitely generated or finitely presented if and only if the associated semigroup has the same property. In some cases, this correspondence will allow us to acquire results about Rees matrix semigroupoids as corollaries of the results in [**6**].

In other cases, however, results about Rees quotients do not appear sufficiently strong to assist with our task, and we must instead construct direct proofs. These proofs follow a similar pattern to those in [**6**], but the nature of semigroupoids introduces some extra subtleties. At a technical level, the partial multiplication in semigroupoids requires that finite graphs and paths substitute for alphabets and words respectively. Often, extra care must be taken to consider whether or not products are defined.

Another difference concerns the relationship between Rees matrix semigroups with and without zero. In the case of semigroups, a Rees matrix semigroup with zero over S can be viewed as a construction over S in its own right, but also as a Rees quotient of a Rees matrix semigroup (without zero) over S with an adjoined zero. In [**6**], the approach taken is first to prove each result for Rees matrix semigroups (without zero). These results are then used, in conjunction with standard results about combinatorial properties of Rees quotients, to extend to Rees matrix semigroups with zero. Adjoining a zero (or rather, zeros) to a semigroupoid is not quite so straightforward. Hence, we choose to construct proofs for the case of Rees matrix semigroups with zero. The simpler case of Rees matrix semigroups (without zero) then arises as a corollary. The need to consider explicitly a zero element introduces some additional complications.

We begin with the following proposition, which gives a sufficient condition for a Rees quotient of a finitely generated or finitely presented semigroup to be respectively finitely generated or finitely presented.

PROPOSITION 4.2. *Let S be a semigroup with zero 0. Let I be an ideal of S, and suppose that $S \setminus I \cup \{0\}$ is a subsemigroup of S. If S is finitely generated [finitely presentable] then the Rees quotient S/I is finitely generated [respectively, finitely presentable].*

PROOF. Let $\langle A \mid R \rangle$ be a presentation for S, and let $\sigma : A^+ \to S$ be a surjective morphism with kernel $R^\#$. Let B be the set of letters in A representing elements of I.

Suppose $w \in A^+$ is a non-empty word *not* representing the zero element $0 \in S$. If w contains a letter in B then, since I is an ideal of S, it follows that w represents an element of I. Conversely, if w does *not* contain a letter in B then all letters in w represent elements of $S \setminus I \cup \{0\}$. Since $S \setminus I \cup \{0\}$ is a subsemigroup of S, and w does not represent 0, it follows that w represents an element of $S \setminus I$. So we have shown that w represents an element of I exactly if it contains a letter in B.

Now let z be a non-empty word in A^+ representing the zero element $0 \in S$. Let

$$R' = R \cup \{(b, z) \mid b \in B\} \subseteq A^+$$

be a new set of relations. Define $\sigma' : A^+ \to S/I$ by

$$w\sigma' = \begin{cases} w\sigma & \text{if } w\sigma \in S \setminus I \\ 0 & \text{otherwise.} \end{cases}$$

Certainly σ' is surjective. We claim that $\langle A \mid R' \rangle$ is a presentation for S/I, with morphism σ.

If $(u,v) \in R$ then $u\sigma = v\sigma$, so $u\sigma' = v\sigma'$ and (u,v) is satisfied in S/I. Furthermore, if $(b,z) \in R' \setminus R$, then $b \in B$ so, $b\sigma \in I$. Hence, $b\sigma' = 0 = z\sigma'$, and (b,z) is satisfied in S/I.

Conversely, suppose a relation $(u,v) \in A^+$ is satisfied in S/I so that $u\sigma' = v\sigma'$. If $u\sigma' = v\sigma' \in S \setminus I$, then $u\sigma = v\sigma$ so that (u,v) is satisfied in S. It follows that u and v are related by a chain of applications of relations in $R \subseteq R'$. On the other hand, if $u\sigma' = v\sigma' = 0$ then $u\sigma, v\sigma \in I$. Now by the argument above, u either represents 0 in S, or contains a letter from B. In the latter case, u is related by an application of a relation of the form (b,z) to a word with z as a factor, which must represent 0 in S. But a word representing 0 in S is necessarily related by a chain of applications of relations in $R \subseteq R'$ to z, so u is related to z. A similar argument shows that v is related by a chain of applications of relations in R' to z, and thence to u, as required.

Thus, we have shown that $\langle A \mid R' \rangle$ is a presentation for S/I. It follows immediately from the method of construction that $\langle A \mid R' \rangle$ is finite [finitely generated] whenever $\langle A \mid R \rangle$ is finite [finitely generated]. $\square$

The following theorem, which gives a sufficient condition for finite generation and finite presentability to be preserved under a Rees matrix construction over a semigroupoid, follows from a straightforward combination of Theorem 3.4, Proposition 4.2, [**6**, Main Theorem] and the reasoning in Section 4.2 above.

THEOREM 4.3. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a semigroupoid S, and let $U = SP'S$ where P' is the set of non-zero entries in the sandwich matrix P. Suppose*

(i) *the indexing sets I and Λ are finite;*
(ii) *the semigroupoid S is finitely generated [finitely presentable]; and*
(iii) *$S \setminus U$ is finite.*

Then M is finitely generated [respectively, finitely presentable].

Proof. Let T be the categorical-at-zero semigroup associated with S, and consider the Rees matrix semigroup $M' = M^0(T; I, \Lambda; P)$. By Theorem 3.4, T is finitely generated [finitely presentable].

Furthermore, we have $TP'T = SP'S \cup \{0\}$, so that $T \setminus TP'T = S \setminus SP'S$ is finite. Let $\overline{T}$ be the semigroup T with a new identity element adjoined. Then clearly $T \setminus \overline{T}P'\overline{T}$ is finite, and it follows by [**6**, Main Theorem], it follows that M' is finitely generated [finitely presentable].

Now by our argument in Section 4.2, M is a subsemigroup of M',

$$(M' \setminus M) \cup \{0\}$$

is an ideal of M' and $M'/((M' \setminus M) \cup \{0\})$ is isomorphic to M. It follows by Proposition 4.2 that M is finitely generated [finitely presentable] as required. □

We now turn our attention to proving a converse for Theorem 4.3. The following proposition gives a necessary condition for a Rees matrix semigroup over a semigroupoid to be finitely generated. It corresponds to the observation in [**6**] that, if a Rees matrix semigroup is finitely generated, the ideal generated by the entries of the sandwich matrix is cofinite in S.

Proposition 4.4. *Suppose $M = M^0(S; F(I), G(\Lambda); P)$ is a finitely generated Rees matrix semigroup with zero over a semigroupoid S. Let*

$$U = SP'S \subseteq S$$

where $P' \subseteq S$ is the set of non-zero entries in the sandwich matrix P. Then U is a subsemigroupoid of S and $S \setminus U$ is finite.

Proof. Clearly, U is closed under the multiplication of S (where defined), and hence is a subsemigroupoid of S.

Now suppose $s \in S \setminus U$. By, Proposition 4.1, we can find $i \in I$ and $\lambda \in \Lambda$ such that $(i, s, \lambda) \in M$. But then (i, s, λ) is clearly indecomposable in

M, so (i, s, λ) must exist in any choice of generators for M. If $S \setminus U$ were infinite, it would follow that any choice of generators for M must be infinite, contradicting the assumption that M is finitely generated. $\square$

The following theorem gives a converse to Theorem 4.3 in the case of finite generation.

THEOREM 4.5. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a semigroupoid S, and let $U = SP'S$ where P' is the set of non-zero entries in the sandwich matrix P. If M is finitely generated then*

(i) *the indexing sets I and Λ are finite;*
(ii) *the semigroupoid S is finitely generated; and*
(iii) *$S \setminus U$ is finite.*

PROOF. Suppose $X \subseteq M$ is a finite set of generators for M. We shall show first that I is finite. To this end, let $i \in I$. Then by Proposition 4.1, there exist $s \in S$ and $\lambda \in \Lambda$ such that $(i, s, \lambda) \in M$. Now by assumption, (i, s, λ) can be written as a product of elements of X. But clearly, the first element of this product must be of the form (i, t, μ). Thus, X contains at least as many elements as I. A similar argument shows that X contains at least as many elements as Λ, and it follows that both I and Λ are finite. Thus, condition (i) holds.[1]

Next, we show that S is finitely generated. Let

$$Z = \{s \in S \mid (i, s, \lambda) \in X \text{ for some } i \in I, \lambda \in \Lambda\} \cup \{P_{\lambda i} \mid \lambda \in \Lambda, i \in I, P_{\lambda i} \neq 0\}.$$

[1]We observe in passing that the simpler argument used in the proof of [**6**, Proposition 2.1] is unsound. If, in their notation, $S \setminus U$ is empty, then $I \times (S \setminus U) \times J$ is finite (indeed, empty) regardless of whether I and J are finite. Their result stands, since an argument akin to ours can easily be substituted.

Clearly, Z is finite. Now for any $s \in S$, by Proposition 4.1 we can find $i \in I$ and $\lambda \in \Lambda$ such that $(i, s, \lambda) \in M$. But M is generated by X, so we have

$$(i, s, \lambda) = (i, s_1, \lambda_1)(i_2, s_2, \lambda_2) \ldots (i_n, s_n, \lambda)$$

for some $s_1, \ldots, s_n \in Z$, $i_2, \ldots, i_n \in I$ and $\lambda_1, \ldots, \lambda_{n-1} \in \Lambda$. Furthermore, because $\{0\}$ is an ideal of M and $(i, s, \lambda) \neq 0$, we must have $P_{\lambda_m i_{m+1}} \neq 0$ for each m with $1 \leq m < n$, so that

$$(i, s, \lambda) = (i, s_1 P_{\lambda_1 i_2} s_2 P_{\lambda_2 i_3} \ldots P_{\lambda_{n-1} i_n} s_n, \lambda).$$

But now we can write

$$s = s_1 P_{\lambda_1 i_2} s_2 P_{\lambda_2 i_3} \ldots P_{\lambda_{n-1} i_n} s_n$$

as a product of elements of Z. Thus, Z generates S, and condition (ii) holds.

Condition (iii) follows directly from Proposition 4.4. □

We now prove a converse to Theorem 4.3, in the case of finite presentability.

THEOREM 4.6. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a semigroupoid S, and let $U = SP'S$ where P' is the set of non-zero entries in the sandwich matrix P. If M is finitely presentable then*

(i) *the indexing sets I and Λ are finite;*
(ii) *the semigroupoid S is finitely presentable; and*
(iii) *$S \setminus U$ is finite.*

PROOF. By Theorem 4.5, we know that conditions (i) and (iii) hold, so we need only to show that condition (ii) holds. Let $B \subseteq M$ be a finite set of generators for M, let

$$Y = \{s \in S \mid (i, s, \lambda) \in B \text{ for some } i \in I, \lambda \in \Lambda\}$$

and let

$$X = (I \times Y \times \Lambda) \cap M.$$

Define a new alphabet

$$A = \{a_{is\lambda} \mid (i, s, \lambda) \in X\} \cup \{z\},$$

where z is a new symbol which will represent the zero element in M. Then there is a natural morphism from A^+ to M which takes z to 0 and each $a_{is\lambda}$ to (i, s, λ). Moveover, since $B \subseteq X \cup \{0\}$ is a set of generators for M, it follows that this morphism is surjective. By Theorem 3.6, it follows that there is a finite presentation $\langle A \mid R \rangle$ for M.

From the proof of Theorem 4.5, we know that

$$Y \cup \{P_{\lambda i} \mid \lambda \in \Lambda, i \in I, P_{\lambda i} \neq 0\}$$

is a finite choice of generators for S. Let C be a graph with vertex set S^0 and edge set

$$\{c_y \mid y \in Y\} \cup \{d_{\lambda i} \mid \lambda \in \Lambda, i \in I, P_{\lambda i} \neq 0\}$$

where $c_y\alpha = y\alpha, c_y\omega = y\omega$, $d_{\lambda i}\alpha = P_{\lambda i}\alpha$ and $d_{\lambda i}\omega = P_{\lambda i}\omega$. Then there is a natural surjective morphism from C^+ onto S, which takes each c_y to y and each $d_{\lambda i}$ to $P_{\lambda i}$. Clearly, C is a finite graph. We seek a finite set R' of relations between paths in C such that $C^+/(R')^\#$ is isomorphic to S.

Let $L \subseteq A^+$ be the set of words in A^+ which do not represent 0, noting that L is closed under the taking of non-empty subwords, and in particular that no words in L contain the letter z. We define mappings $\phi : L \to C^+$, $\mu : L \to I$ and $\nu : L \to \Lambda$ by

$$(a_{i_1 y_1 \lambda_1} \ldots a_{i_n y_n \lambda_n})\phi = c_{y_1} d_{\lambda_1 i_2} c_{y_2} d_{\lambda_2 i_3} \ldots c_{y_n},$$

$$(a_{i_1 y_1 \lambda_1} \ldots a_{i_n y_n \lambda_n})\mu = i_1$$

and

$$(a_{i_1 y_1 \lambda_1} \ldots a_{i_n y_n \lambda_n})\nu = \lambda_n.$$

Clearly, $w \in L$ represents $(i, x, \lambda) \in M$ exactly if $w\mu = i$, $w\nu = \lambda$ and $w\phi \in C^+$ represents $x \in S$.

Let $x \in S$ be arbitrary. By Proposition 4.1 there exist $i \in I$ and $\lambda \in \Lambda$ such that $(i, x, \lambda) \in M$. Now since M is generated by A, we can choose some word $w \in L$ representing (i, x, λ). But now $w\phi \in C^+$ represents $x \in S$. We have shown that every arrow of S is represented by a path in $L\phi$.

Now for each $w \in C^+$ of length no more than 2, choose some path $w\chi$ in $L\phi$ representing the same element as w. We define a set $R' \subseteq C^+ \times C^+$ of relations by

$$\begin{aligned} R' &= \{(w, w\chi) \mid w \in C^+ \text{ has length no more than } 2\} && (6) \\ &\cup \{(u\phi, v\phi) \mid (u, v) \in R \cap (L \times L)\}. && (7) \end{aligned}$$

Observe that R' is finite. We claim that $\langle C \mid R' \rangle$ is a presentation for S, with the obvious isomorphism. It follows easily from the definitions of χ and ϕ that all relations in R' are satisfied in S. It remains to show that every relation satisfied in S is a consequence of the relations in R'. Our strategy is to show first that every path in C^+ is related as a consequence of the relations in R' to a path in $L\phi$, and secondly that any relation between paths in $L\phi$ which holds in S is a consequence of relations in R'.

Suppose $w = w_1 w_2 \dots w_n \in C^+$, with each $w_i \in C$. If $n \leq 2$ then $(w, w\chi) \in R'$ and $w\chi \in L\phi$. Now assume for induction that $n > 2$ and that every w of length no more than $n - 1$ is related by a relation in $(R')^\#$ to a path in $L\phi$. Then $w_2 \dots w_n (R')^\# w'$ for some $w' \in L\phi$, say

$$w' = (a_{i_1 x_1 \lambda_1} \dots a_{i_m x_m \lambda_m})\phi = c_{x_1} d_{\lambda_1 i_2} c_{x_2} \dots d_{\lambda_{m-1} i_m} c_{x_m} \in C^+$$

where $a_{i_1 x_1 \lambda_1} \dots a_{i_m x_m \lambda_m} \in L$. Now

$$w_1 \alpha = w_2 \omega = w' \omega = c_{x_1} \omega$$

so that $w_1 c_{x_1}$ is a path in C^+. Certainly $(w_1 c_{x_1})\chi \in L\phi$, say

$$(w_1 c_{x_1})\chi = (a_{j_1 y_1 \mu_1} \dots a_{j_k y_k \mu_k})\phi$$

where $a_{j_1y_1\mu_1}\dots a_{j_ky_k\mu_k} \in L$.

Since $a_{j_ky_k\mu_k} \in A$, we deduce that $y_k \in Y$, and that $j_kF = y_k\omega$. Furthermore, we have

$$\lambda_1 G = x_1\alpha = c_{x_1}\alpha = w_1 c_{x_1}\alpha = y_k\alpha.$$

It follows from the definitions of X and A that $(j_k, y_k, \lambda_1) \in X$, so there is a letter $a_{j_ky_k\lambda_1} \in A$. Furthermore, we have that $P_{\mu_{k-1}j_k}$ and $P_{\lambda_1 i_2}$ are non-zero, so we can deduce that there is a word

$$a_{j_1y_1\mu_1}\dots a_{j_ky_k\lambda_1}a_{i_2x_2\lambda_2}\dots a_{i_mx_m\lambda_m} \in L.$$

By the definition of ϕ we have

$$(a_{j_1y_1\mu_1}\dots a_{j_ky_k\lambda_1}a_{i_2x_2\lambda_2}\dots a_{i_mx_m\lambda_m})\phi = c_{y_1}d_{\mu_1j_2}\dots c_{y_k}d_{\lambda_1i_2}\dots d_{\lambda_{m-1}i_m}c_{x_m}.$$

But now

$$\begin{aligned} w \quad (R')^{\#} &\quad w_1w' \\ &= w_1c_{x_1}d_{\lambda_1i_2}c_{x_2}\dots d_{\lambda_{m-1}i_m}c_{x_m} \\ (R')^{\#} &\quad (w_1c_{x_1})\chi d_{\lambda_1i_2}c_{x_2}\dots d_{\lambda_{m-1}i_m}c_{x_m} \\ &= c_{y_1}d_{\mu_1j_2}\dots c_{y_k}d_{\lambda_1i_2}\dots d_{\lambda_{m-1}i_m}c_{x_m} \\ &\in L\phi. \end{aligned}$$

We have shown that every path in C^+ is related by $(R')^{\#}$ to a path in $L\phi$.

Now suppose $y, y' \in L$ are such that the relation $(y\phi, y'\phi)$ is satisfied in S, that is, such that $y\phi$ and $y'\phi$ represent the same arrow in S. We shall show that $y\phi$ and $y'\phi$ are related as a consequence of relations in R'. Suppose

$$y = a_{iy_1\lambda_1}a_{i_2y_2\lambda_2}\dots a_{i_my_m\lambda} \in L$$

and

$$y' = a_{i'y'_1\lambda'_1}a_{i'_2y'_2\lambda'_2}\dots a_{i'_ny'_n\lambda'} \in L.$$

Consider the word

$$y'' = a_{iy_1'\lambda_1'}a_{i_2'y_2'\lambda_2'}\dots a_{i_n'y_n'\lambda} \in L,$$

observing that $y'\phi = y''\phi$. Then $y\phi$ and $y'\phi = y''\phi$ represent the same arrow in S, and furthermore, $y\mu = y''\mu$ and $y\nu = y''\nu$. By our observation above, we deduce that y and y'' represent the same element of M.

Now since $< A \mid R >$ is a presentation for M, we know that y and y'' are related as a consequence of R. Suppose first that they are related by a single application of a relation in R, that is, that $y = sut$ and $y'' = svt$ for some $s, t \in A^*$ and some relation $(u, v) \in R$ or $(v, u) \in R$. Furthermore, since $y, y' \in L$ and L is closed under the taking of non-empty subwords, we have $u, v \in L$.

Then from the definition of ϕ, we have

$$y\phi = (s\phi)(d_{(s\nu)(u\mu)})(u\phi)(d_{(u\nu)(t\mu)})(t\phi)$$

and similarly

$$y'\phi = y''\phi = (s\phi)(d_{(s\nu)(v\mu)})(v\phi)(d_{(v\nu)(t\mu)})(t\phi).$$

But since the relation (u, v) holds in M, we have $u\mu = v\mu$ and $u\nu = v\nu$, so that $d_{(s\nu)(u\mu)} = d_{(s\nu)(v\mu)}$ and $d_{(u\nu)(t\mu)} = d_{(v\nu)(t\mu)}$. It follows immediately that $y\phi$ and $y'\phi$ are related by an application of a relation $(u\phi, v\phi)$ or $(v\phi, u\phi)$ from (7).

Now suppose $y, y' \in L$ and $y = y_1, y_2, y_3, \dots, y_n = y'$ are such that each y_i is related to y_{i+1} by an application of a relation in R. Then since relations in R are satisfied in M, each y_i represents the same non-zero element of M, and so each $y_i \in L$. It follows that each $y_i\phi$ is related to $y_{i+1}\phi$ by an application of a relation from (7), so $y\phi$ is related as a consequence of R' to $y'\phi$.

This completes the proof. $\square$

4.4. Rees Matrix Semigroups Without Zero

Our results about Rees matrix semigroups with zero adapt easily to the case of Rees matrix semigroups without zero.

THEOREM 4.7. *Let $M = M(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup (without zero) over a semigroupoid S, and let $U = SP'S$ where P' is the set of entries in the sandwich matrix P. Then M is finitely generated [finitely presentable] if and only if*

(i) *the indexing sets I and Λ are finite;*
(ii) *the semigroupoid S is finitely generated [respectively, finitely presentable]; and*
(iii) *$S \setminus U$ is finite.*

PROOF. Consider the Rees matrix semigroup with zero

$$M' = M^0(S; F(I), G(\Lambda); P).$$

Then $M = M' \setminus \{0\}$ is a cofinite subsemigroup of M'. By Theorem 3.5 (or Theorem 1.1 [Theorem 1.3] of [**62**]), the semigroup M is finitely generated [finitely presentable] if and only if M' is finitely generated [finitely presentable]. By Theorem 4.3 together with Theorem 4.5 [Theorem 4.6], this is so if and only if conditions (i), (ii) and (iii) hold. □

A combination of Theorems 4.3, 4.5 and 4.7 yields the following additional implication.

COROLLARY 4.8. *Suppose M is a finitely generated Rees matrix semigroup (with or without zero) over a finitely presentable semigroupoid S. Then M is finitely presentable.*

CHAPTER 5

Regularity

In this chapter, we focus on the notions of *regularity* and *synchronous regularity*. We begin by recapping the usual definitions of finite automata and regular languages over a given alphabet, and of synchronous regularity. We then introduce analogous constructs over finite graphs, in place of alphabets. It is not strictly necessary to take this approach, since it would be possible to do everything we require by considering certain subsets of more usual languages over alphabets. Indeed, we shall prove that the two approaches lead to equivalent definitions. However, from a conceptual standpoint, semigroupoids are perhaps more naturally viewed as homomorphic images of free semigroupoids, rather than partial homomorphic images of free semigroups.

We also state and prove a number of technical lemmas, which give sufficient conditions for functions to preserve regularity and synchronous regularity in languages over graphs (and hence also in languages over alphabets). These lemmas allow us to avoid detailed technical consideration of automata in the chapters that follow. An alternative strategy would involve the deployment of finite transducers (or generalised sequential machines).

Some of the material in this chapter previously appeared in the preprint [**44**].

5.1. Languages and Automata

Let A be a finite *alphabet*, or set of symbols. A *word* over A is a finite sequence of zero or more symbols in A. The set of all words over A forms a monoid under the operation of *concatenation*, called the free monoid on A and denoted A^*. The set of non-empty words over A forms a semigroup, called the free semigroup on A and denoted A^+.

Recall that a *language* over A is a set of words over A, that is, a subset of the free monoid A^*. A *(finite) automaton* over A is a (finite) graph, with a set of distinguished *start vertices*, a set of distinguished *terminal vertices*, and edges labelled by elements of A. The vertices of an automaton are traditionally called *states*, while the labelled edges are called *transitions*.

The labelling of the edges by elements of A induces a natural labelling of the paths by words over A, that is, by words in A^*. We say that a word over A is *accepted* by the automaton if it labels a path from a start vertex to a terminal vertex. The (unique) *language accepted* (or *language recognised*) by the automaton is the set of all words over A which are accepted by the automaton.

A language over A is called *regular* if it is recognised by some finite automaton. For an introduction to the theory of regular languages and finite automata, the reader is referred to one of the many textbooks on this subject, such as [**39**].

5.2. From Alphabets to Graphs

Let X be a finite graph. A *path language* or just a *language* over X is a (possibly empty) collection of (possibly empty) paths in X, that is, a subset of the free category X^*.

If X is a finite graph and $N \in \mathbb{N}$, we denote by X^N the language of paths in X of length N. Recall from Section 2.4 that we identify each path of length 1 with its single edge, and each vertex with the empty path at that vertex, so that X^1 and X^0 are consistent with their definitions in Chapter 2. Given $N \in \mathbb{N}$, we write

- $X^{\geq N}$ for the language of paths in X^* of length N or more;

- $X^{>N}$ for the language of paths in X^* of length strictly greater than N;
- $X^{\leq N}$ for the language of paths in X^* of length N or less; and
- $X^{<N}$ for the language of paths in X^* of length strictly less than N.

Note that these definitions include empty paths, where appropriate. Given a path $w \in X^*$, we denote by $|w|$ the length of the path w.

Let $x = x_1 \ldots x_n$ be a path in a free semigroupoid X^+ where each $x_i \in X^1$. A *prefix* of x is a (non-empty) path of the form $x_1 \ldots x_j$ for some $1 \leq j \leq n$. A *suffix* of x is a (non-empty) path of the form $x_j \ldots x_n$ for some $1 \leq j \leq n$. A *factor* of x is a (non-empty) path of the form $x_j \ldots x_k$ for some $1 \leq j \leq k \leq n$. An *internal factor* of x is a (non-empty) path of the form $x_j \ldots x_k$ for some $1 < j \leq k < n$, that is, a factor of $x_2 \ldots x_{n-1}$.

Given a path language $L \subseteq X^+$, we write

- $\mathrm{Pref}(L)$ for the set of non-empty prefixes of paths in L;
- $\mathrm{Suff}(L)$ for the set of non-empty suffixes of paths in L;
- $\mathrm{Fact}(L)$ for the set of non-empty factors of paths in L; and
- $\mathrm{Int}(L)$ for the set of non-empty internal factors of paths in L.

For $N \in \mathbb{N}$, we also define

- $\mathrm{Pref}_N(L) = \mathrm{Pref}(L) \cap X^N$;
- $\mathrm{Suff}_N(L) = \mathrm{Suff}(L) \cap X^N$;
- $\mathrm{Fact}_N(L) = \mathrm{Fact}(L) \cap X^N$; and
- $\mathrm{Int}_N(L) = \mathrm{Int}(L) \cap X^N$,

noting that all four are empty when $N = 0$.

We observe that there is a natural correspondence between alphabets, and finite graphs with a single vertex, each letter corresponding to an edge. The free semigroup [monoid] on the alphabet is the unique local semigroup of the free semigroupoid [respectively, category] on the corresponding single-vertex graph.

5.3. Automata over Graphs

A *(path) automaton* M over a small graph X is a small graph M, together with a graph morphism $\sigma : M \to X$, a set of distinguished *start vertices* of M and a set of distinguished *terminal vertices* of M.

We think of the automaton as the graph M with each edge $e \in M^1$ labelled by $e\sigma^1 \in X^1$, and each vertex $v \in M^0$ labelled by $v\sigma^0 \in X^0$. Again, the vertices and edges are called *states* and *transitions* respectively. The unique *language accepted* or *language recognised* by M is the set of paths in X which label paths from a start vertex to a terminal vertex in M.

The automaton is called a *complete, deterministic* automaton if

(i) there is exactly one start vertex in the pre-image of each object in X; and

(ii) for every state $m \in M^0$ and every edge $e \in X^1$ with $e\omega = m\sigma^0$ there is a unique edge $f \in M^1$ with $f\sigma^1 = e$.

The automaton is called *finite* if the graphs M and X are finite. We shall be concerned exclusively with finite automata. A path language which is accepted by some finite path automaton is called *regular*.

Example 5.1. Figure 1 illustrates a simple example of a finite path automaton M over a graph X, with edge-morphism $\sigma : M \to X$. The labels on the vertices and edges of X may be thought of as names for those edges and vertices. The labels on the states and transitions of M indicate the respective vertices and edges of X to which they map. M has one start state, indicated by an arrow coming in with no source, and one terminal state, indicated by a double circle.

The automaton is finite, since M and X are both finite graphs. However, it is not a complete, deterministic automaton. The path language accepted is the set $\{aw \mid w \in B^1\}$, where B is the subcategory of X^* generated by the set $\{bc, b'c'\}$.

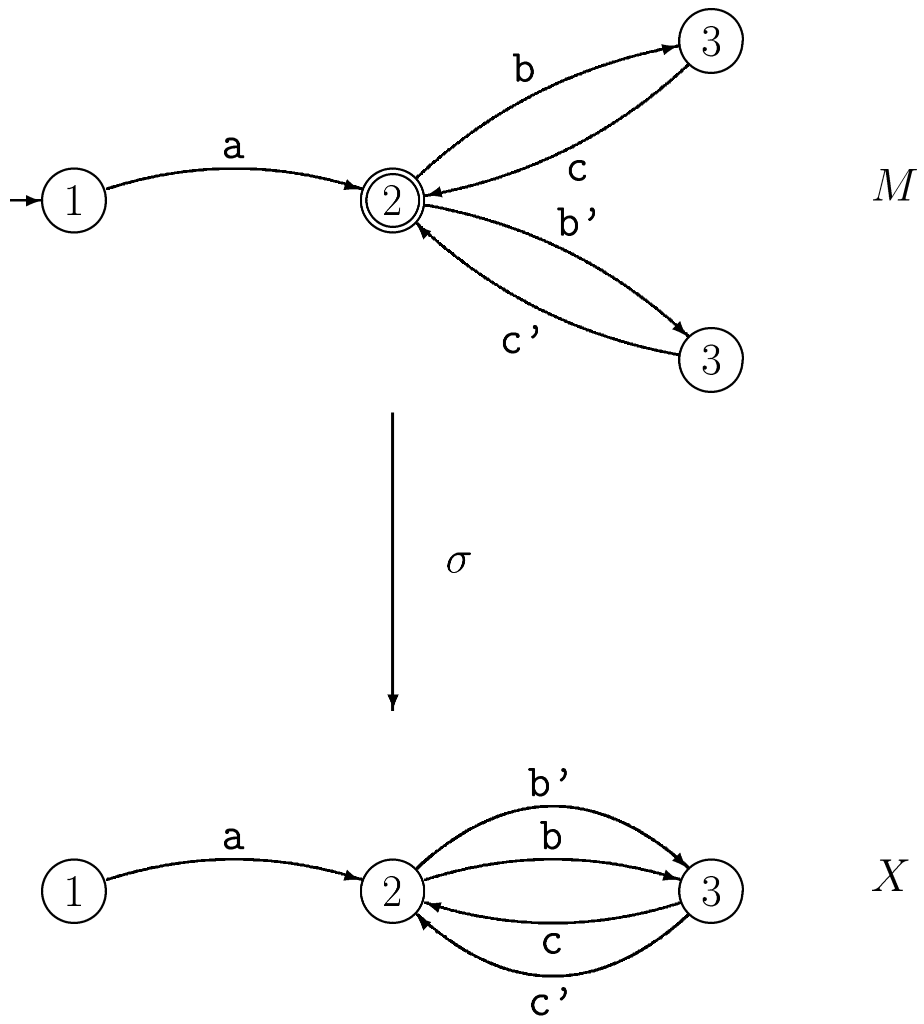

FIGURE 1. A finite path automaton M over a graph X.

Note that non-empty paths in a finite graph X are defined to be sequences of edges from X^1, and as such are identified with the corresponding words over X^1 when viewed as an alphabet. Thus, a language of non-empty paths over X is also a language (in the more usual sense) over X^1. The following proposition tells us that there is no ambiguity in describing such a language as regular.

PROPOSITION 5.2. *Let X be a finite graph and L be a set of non-empty paths in X. Then L is a regular path language over X if and only if L is a regular language over the set X^1 of edges of X.*

PROOF. If L is a regular path language over X, there is a finite path automaton M over X which accepts L. We can view M as an automaton over the alphabet X^1, by disregarding vertex labels. It is easy to verify that the automaton still accepts the language L, so L is a regular language over X^1.

Conversely, suppose L is a regular language over X^1, which contains only non-empty words which are identified with paths in X, so that L is also a path language over X. Then L is accepted by some complete, deterministic finite automaton M over X^1. We define a finite graph automaton N over X with

- vertex set $M^0 \times X^0$, where each (m, x) is labelled x;
- start vertices (q_0, x) where q_0 is the start vertex of M and $x \in X^0$;
- terminal vertices (q, x) such that q is a terminal vertex in M and $x \in X^0$; and
- an edge from $(q, s\omega)$ to $(q', s\alpha)$ with label s whenever M has an edge from q to q' with label s.

It is an easy exercise to verify that N is a finite path automaton accepting L. $\square$

Proposition 5.2 completely characterises those regular path languages which contain only non-empty paths. Recall that we identify each vertex $x \in X^0$ with the empty path at that vertex in X^*. Since the graph X has finitely many vertices, the following proposition tells us that the presence or absence of empty paths does not affect regularity of a path language.

PROPOSITION 5.3. *Let L be a language over a graph X, and $x \in X^0$ a vertex of X. Then $L \setminus \{x\}$ is regular if and only if $L \cup \{x\}$ is regular.*

PROOF. Suppose $L \setminus \{x\}$ is regular. Let M be an automaton accepting $L \setminus \{x\}$. Let q' be a symbol not in M^0. We construct from M a new finite automaton N with

- state set $M^0 \cup \{q'\}$ with states in M^0 labelled as in M, and q' labelled x;
- terminal states those of M, plus q';
- start states those of M except those labelled by x, which are replaced by q';
- all the edges of M; and
- an additional edge from q' to p with label $y \in X^1$ whenever $y\omega = x$ and M has an edge from a start state q to p with label y.

Then it is easy to verify that N accepts the path language $L \cup \{x\}$, so that $L \cup \{x\}$ is regular.

Conversely, if $L \cup \{x\}$ is a regular language, then let M be a path automaton accepting it. Construct a new automaton N from M exactly as above, except that the new state q' is not a terminal state. Then it is easy to verify that N accepts $L \setminus \{x\}$, so that $L \setminus \{x\}$ is regular. $\square$

It is well-known that every regular language over an alphabet is accepted by a complete, deterministic finite automaton. Unsurprisingly, this result also holds for regular path languages and finite path automata.

PROPOSITION 5.4. *Let $L \subseteq X^*$ be a language over a finite graph X. Then L is a regular path language over X if and only if L is accepted by a complete, deterministic, finite path automaton over X.*

PROOF. Suppose L is a regular path language over X. It follows from Proposition 5.3 that $L \cap X^+$ is also a regular path language over X and by Proposition 5.2 that $L \cap X^+$ is a regular language over X^1. By a standard result about regular languages, we can choose a complete, deterministic, finite automaton M over X^1 which accepts $L \cap X^+$.

We construct from M a new automaton N over X recognising $L \cap X^+$, as in the proof of Proposition 5.2. It follows immediately from the method of construction that N is a complete, deterministic automaton. We now apply the method used in the proof of Proposition 5.3 to obtain from N a new automaton accepting L. Again, the method of construction ensures that this is a complete, deterministic automaton, as required.

The converse is immediate. $\square$

Another important result is the following.

PROPOSITION 5.5. *Let X be a finite graph. Then X^* is a regular path language over X.*

PROOF. We view X as an automaton over itself, with $\sigma : X \to X$ the identity graph morphism and every state a start and a terminal state. Then every path in X^* labels a path in X^* (namely, itself), and that path must run from a start state to a terminal state. Thus, the language accepted by X is the whole of X^*. $\square$

A consequence of Propositions 5.2, 5.3 and 5.5 is that a large number of standard facts about regular languages extend immediately to regular path languages.

LEMMA 5.6. *Let X be a finite graph. Then:*

- (i) *All finite path languages over X are regular.*
- (ii) *The free semigroupoid X^+ is a regular path language.*
- (iii) *Regular path languages over X are closed under the operations of concatenation, finite intersection, finite union, complement, set difference and generation of subcategories and subsemigroupoids.*

(iv) *If L is a regular path language over X, then* $\mathrm{Pref}(L)$ *is a regular path language over X.*

We shall make free use of the properties given by Lemma 5.6 without explicit reference.

5.4. Synchronous Regularity

Given a graph X, we denote by $X^\$$ the graph X with an extra edge $\$_s$ adjoined for every vertex $s \in X^0$, with source and target s. We define a function

$$\delta_X : (X^+ \times X^+) \to (X^\$ \times X^\$)^+$$

by

$$(a,b)\delta_X = \begin{cases} (a_1,b_1)\dots(a_n,b_n) & \text{if } m = n \\ (a_1,b_1)\dots(a_n,b_n)(a_{n+1},\$_{b_n\alpha})\ \dots(a_m,\$_{b_n\alpha}) & \text{if } m > n \\ (a_1,b_1)\dots(a_m,b_m)(\$_{a_m\alpha},b_{n+1})\ \dots(\$_{a_m\alpha},b_n) & \text{if } n > m \end{cases}$$

where $a = a_1 \dots a_m$ and $b = b_1 \dots b_n$.

Where only one graph X features in our discussion, we shall for readability write δ in place of δ_X. We observe that the function δ_X is injective, and in particular that it distributes over intersection, that is, that

$$(R_1 \cap R_2)\delta_X = R_1\delta_X \cap R_2\delta_X$$

for all binary relations $R_1, R_2 \subseteq X^+ \times X^+$.

A *(binary) synchronous path automaton* over X is a finite automaton over the graph $X^\$ \times X^\$$, with the property that the language accepted is contained within the image $(X^+ \times X^+)\delta_X$ of δ_X. For convenience, we shall say that a synchronous path automaton accepts a pair

$$(a,b) \in X^+ \times X^+$$

if it accepts $(a,b)\delta_X$. A binary relation $R \subseteq X^+ \times X^+$ is called *(right) synchronously regular* if there exists a synchronous path automaton accepting exactly the language $R\delta_X$, that is, if $R\delta_X$ is regular.

We emphasise that we have defined synchronous regularity to be a property of binary relations on free semigroupoids, and not of sequences of pairs of letters. This slightly more abstract approach allows us to avoid cluttering the exposition in later chapters with the technical details associated with synchronisation.

We could equally naturally use a function, in place of δ, which pads the shorter path on the *left-hand* side, instead of the right-hand-side. This leads to a dual notion of *left* synchronous regularity. We shall be primarily concerned with right synchronous regularity, which we shall usually call just synchronous regularity.

Just as a language of non-empty paths over X is also a language over X^1, so a binary relation on X^+ is also a binary relation on $(X^1)^+$. Just as Proposition 5.2 told us that there was no ambiguity in describing a path language as regular, the following proposition ensures that there is no ambiguity in describing a binary relation on a finitely generated free semigroupoid as synchronously regular.

PROPOSITION 5.7. *Let X be a finite graph, and $R \subseteq X^+ \times X^+$. Then R is synchronously regular as a subset of $X^+ \times X^+$ if and only if R is synchronously regular as a subset of $(X^1)^+ \times (X^1)^+$.*

PROOF. If R is synchronously regular as a subset of $X^+ \times X^+$ then, by definition, $R\delta_X$ is a regular language over $(X^\$ \times X^\$)^+$. Let M be an automaton over $X^\$ \times X^\$$ accepting $R\delta_X$. We obtain from M a new automaton N over $(X^1)^\$ \times (X^1)^\$$ by replacing each label of the form $(x, \$_s)$ with a label $(x, \$)$, and each label of the form $(\$_s, x)$ with a label $(\$, x)$, and disregarding labels of vertices. Then it is easy to verify that N accepts $R\delta_{X^1}$, so that R is synchronously regular as a subset of $(X^1)^+ \times (X^1)^+$.

Conversely, if $R \subseteq X^+ \times X^+$ is synchronously regular as a subset of $(X^1)^+ \times (X^1)^+$ then there exists a synchronous automaton M over $(X^1)^\$ \times (X^1)^\$$ recognising R. We obtain from M a new automaton N over $X^\$ \times X^\$$, which operates like M but keeps track of the vertices reached while following the paths read in the graph X. We define the automaton N to have

- state set $M^0 \times X^0 \times X^0$, where each (m, x, y) is labelled (x, y);
- start states (q_0, x, y) where q_0 is a start state of M and $x, y \in X^0$;
- terminal states (q, x, y) where q is a terminal state of M, and $x, y \in X^0$;
- an edge from $(q, x\omega, y\omega)$ to $(q', x\alpha, y\alpha)$ with label (x, y) whenever M has an edge from q to q' with label (x, y);
- for every $t \in X^0$, an edge from $(q, x\omega, t)$ to $(q', x\alpha, t)$ with label $(x, \$_t)$ whenever M has an edge from q to q' with label $(x, \$)$; and
- for every $t \in X^0$, an edge from $(q, t, y\omega)$ to $(q', t, y\alpha)$ with label $(\$_t, y)$ whenever M has an edge from q to q' with label $(\$, y)$.

Once again, it is readily verified that N accepts $R\delta_X$, so that R is synchronously regular as a subset of $X^+ \times X^+$. □

Just as we were able to generalise to regular path languages a number of standard facts about regular languages, so Proposition 5.7 lets us generalise to synchronously regular path languages some results about synchronously regular languages. The following properties are all generalisations of standard results about synchronously regular languages, which can be found in [**26**].

LEMMA 5.8. *Let X be a finite graph. Then:*

(i) *If $R \subseteq X^+ \times X^+$ is synchronously regular, then the binary relation*

$$R^{-1} = \{(v, u) \mid (u, v) \in R\}$$

is synchronously regular.

(ii) *If $R \subseteq X^+ \times X^+$ is synchronously regular, then the projections $R\pi_1$ and $R\pi_2$ of R onto its first and second coordinates are regular.*

(iii) *If $K, L \subseteq X^+$ are regular languages of non-empty paths, then the binary relation $K \times L$ is synchronously regular.*

(iv) *Synchronously regular binary relations over X are closed under finite union, finite intersection and relational composition.*

(v) *If $K \subseteq X^+$ is regular then the diagonal binary relation*

$$\{(w, w) \mid w \in K\}$$

is synchronously regular.

5.5. Sliding Windows

Let X and Y be finite graphs, $A \subseteq X^*$ and $\phi : A \to Y^*$ be a function. We say that ϕ is *regularity-preserving* if for every regular language $L \subseteq A \subseteq X^+$ contained in A, we have that $L\phi \subseteq Y^+$ is a regular language. We say that ϕ is *strongly regularity-preserving* if for every regular language $L \subseteq X^+$, we have that $(L \cap A)\phi \subseteq Y^+$ is a regular language. Note that a regularity-preserving function with a regular domain is necessarily strongly regularity-preserving.

A *sliding window inverse* for a function $\phi : A \to Y^+$ is a quadruple (n, f, g, h) consisting of a positive integer n and three functions

$$f : \mathrm{Pref}_n(A\phi \cap Y^{>n}) \to X^*,$$

$$g : \mathrm{Int}_n(A\phi) \to X^*$$

and

$$h : \mathrm{Suff}_n(A\phi \cap Y^{>n}) \to X^*,$$

with the property that for any $w \in A$ and $y_1 \dots y_k \in Y^+$ with $y_1, \dots, y_k \in Y$ and $k > n$ such that $w\phi = y_1 \dots y_k$, we have

$$w = (y_1 \dots y_n)f(y_2 \dots y_{n+1})g(y_3 \dots y_{n+2})g \ \dots (y_{k-n} \dots y_{k-1})g(y_{k-n+1} \dots y_k)h.$$

If f, g and h are functions with domains containing those given above, we shall say that (n, f, g, h) is a sliding window inverse for ϕ if the restrictions of f, g and h to the appropriate domains have the given properties.

Intuitively, a function $\phi : A \to Y^+$ has a sliding window inverse if, given a sufficiently long path $w\phi$, one can slide a "window" of finite length from left to right along $w\phi$, and reconstruct w from left to right, using at each stage only the information visible in the "window". Since w is reconstructed from left to right, it can be fed directly into a finite automaton over X, allowing one to test whether w belongs to a given regular language L. This process can be simulated by a finite automaton, which reads $w\phi$ one character at a time, stores (encoded in its state) the last n characters read and the state which the automaton over X would have reached, having read the prefix of w so far constructed.

This process makes no sense with an arbitrary word $w' \in Y^+$, so it is also necessary that the automaton be able to check that w' arises in the form $w\phi$, that is, that $w' \in A\phi$. Thus, we require that ϕ have a regular image $A\phi$. With this proviso, it then follows that functions with sliding window inverses are strongly regularity-preserving. This argument is formalised in the following lemma.

Lemma 5.9 (The Sliding Window Lemma). *Let X and Y be finite graphs, $A \subseteq X^*$ and $\phi : A \to Y^*$ be a function. If $A\phi \subseteq Y^*$ is regular, and ϕ has a sliding window inverse, then ϕ is strongly regularity-preserving.*

Proof. Let (n, f, g, h) be a sliding window inverse for ϕ, and suppose $L \subseteq X^+$ is a regular language. Let M be a complete, deterministic finite path automaton accepting L. Clearly, the set $(L \cap A)\phi \cap Y^{\leq 2n}$ is finite, and hence regular, so we concern ourselves only with the set $(L \cap A)\phi \cap Y^{>2n}$. Moreover, this set is a finite union of the form

$$(L \cap A)\phi \cap Y^{>2n} = \bigcup_{a,b \in Y^n} L_{a,b}$$

where each

$$L_{a,b} = (L \cap A)\phi \cap aY^+b.$$

Since the class of regular path languages is closed under finite union, we need only show that $L_{a,b}$ is regular for each $a, b \in Y^n$. To this end, let $a, b \in Y^n$ be fixed, and suppose $a = a_1 \dots a_n$ and $b = b_1 \dots b_n$.

We construct a new automaton N over Y with

- state set $M^0 \times Y^{n-1}$, where each (m, w) is labelled $w\omega$;
- start state $(q_0', a_2 \dots a_n)$, where q_0' is the unique state in M^0 such that M has a path from a start state to q_0' labelled $(a)f$;
- terminal states $(q, b_1 \dots b_{n-1})$ such that M has a path from q to some terminal state with label $(b)h$; and
- for each $x_1 \dots x_n \in Y^n$, an edge from $(q, x_1 \dots x_{n-1})$ to $(q', x_2 \dots x_n)$ with label x_n whenever M has a path from q to q' with label $(x_1 \dots x_n)g$.

Let K be the language accepted by N. We claim that

$$L_{a,b} = aKb_n \cap A\phi \cap aY^+b.$$

Since K, $A\phi$ and aY^+b are regular, it will then follow that $L_{a,b}$ is regular, as required.

Certainly we have

$$L_{a,b} \subseteq A\phi \cap aY^+b$$

so it will suffice to show that whenever $w \in Y^+$ is such that $awb_n \in A\phi \cap aY^+b$, we have $w \in K$ if and only if $awb_n \in L_{a,b}$. To this end, suppose $w \in Y^+$ is such that $awb_n \in A\phi \cap aY^+b$. Let $w_0, \dots, w_k \in Y$ be such that $awb_n = w_0 \dots w_k$, so that $w = w_n \dots w_{k-1}$. Let

$$w' = (a)f(w_1 \dots w_n)g(w_2 \dots w_{n+1})g \dots (w_{k-n} \dots w_{k-1})g(b)h.$$

From the definition of a sliding window inverse, w' is the unique word in A such that $w'\phi = awb_n$. We shall show that $awb_n \in L_{a,b}$ if and only if w is accepted by N.

Now $awb_n \in L_{a,b}$ if and only if

$$awb_n \in (L \cap A)\phi \cap Y^{>2n}.$$

But since we know that $awb_n \in A\phi \cap Y^{>2n}$, and w' is the unique word in A with $w'\phi = awb_n$, this is the case if and only if $w' \in L$, that is, if and only if w' is accepted by M. Now this is true exactly if M has states $q_1, q_2, \ldots, q_{k-n}$ such that:

(i) M has a path from a start state to q_1, labelled $(a)f$;

(ii) for $1 \leq i < k-n$, M has a path from q_i to q_{i+1} labelled $(w_i \ldots w_{i+n-1})g$; and

(iii) M has a path from q_{k-n} to some terminal state labelled $(b)h$.

But, by the definition of N, this is the case exactly if M has states $q_1, q_2, \ldots, q_{k-n}$ such that:

(i) $(q_1, w_1 \ldots w_{n-1})$ is the start state in N;

(ii) for $1 \leq i < k-n$, N has an edge from $(q_i, w_i \ldots w_{i+n-2})$ to $(q_{i+1}, w_{i+1} \ldots w_{i+n-1})$ labelled w_{i+n-1};

(iii) $(q_{k-n}, w_{k-n+1} \ldots w_{k-1})$ is a terminal state in N.

This in turn is true exactly if N has a path from the start state to some terminal state $(q_{k-n}, w_{k-n+1} \ldots w_{k-1})$ with label $w_n \ldots w_{k-1} = w$. But this is the case exactly if w is accepted by N, that is, exactly if $w \in K$.

This completes the proof that $(L \cap A)\phi$ is regular, justifying the claim that ϕ is strongly regularity-preserving. $\square$

5.6. Synchronous Sliding Windows

Let X and Y be finite graphs, and for $i = 1, 2$ suppose we have subsets $A_i \subseteq X^+$ and functions $\phi_i : A_i \to Y^+$. We say that ϕ_1 and ϕ_2 are *mutually synchronous regularity preserving* if for every synchronously regular binary relation $R \subseteq A_1 \times A_2$, the binary relation

$$\{(u\phi_1, v\phi_2) \mid (u, v) \in R\} \ \subseteq Y^+ \times Y^+$$

is synchronously regular. We say that ϕ_1 and ϕ_2 are *strongly mutually synchronous regularity-preserving* if for every synchronously regular relation $R \subseteq X^+ \times X^+$, the relation

$$\{(u\phi_1, v\phi_2) \mid (u, v) \in R \cap (A_1 \times A_2) \ \} \subseteq Y^+ \times Y^+$$

is synchronously regular.

If $A \subseteq X^+$ and $\phi : A \to Y^+$ is (strongly) mutually synchronous regularity preserving with itself, we say that ϕ is *(strongly) synchronous regularity preserving*.

Let $\psi : \mathbb{N} \to \mathbb{N}$ be a function. We say that sliding window inverses (n_1, f_1, g_1, h_1) for ϕ_1 and (n_2, f_2, g_2, h_2) for ϕ_2 are *synchronised by* ψ if

(i) $n_1 = n_2$;

(ii) for every $i = 1, 2$ and every $w \in \mathrm{Pref}_{n_1}(A_i\phi_i \cap Y^{>n_1})$, we have $|wf_i| = 0\psi$; and

(iii) for every $i = 1, 2$ and every $xyz \in A_i\phi_i$ with $x \in Y^m$, $m \geq 1$, $y \in Y^{n_1}$ and $z \in Y^+$ we have $|yg_i| = m\psi$.

We say that two sliding window inverses are *synchronised* if they are synchronised by some function $\psi : \mathbb{N} \to \mathbb{N}$. Note that a sliding window inverse need not be synchronised with itself. A sliding window inverse which is synchronised with itself, we call *self-synchronised*.

The following lemma gives a sufficient condition for two injective functions to be strongly mutually synchronous regularity preserving.

LEMMA 5.10 (The Synchronous Sliding Windows Lemma). *Let X and Y be finite graphs, $A_1, A_2 \subseteq X^+$ and $\phi_1 : A_1 \to Y^+$ and $\phi_2 : A_2 \to Y^+$ be injective functions. If $A_1\phi_1$ and $A_2\phi_2$ are regular and ϕ_1 and ϕ_2 have synchronised sliding window inverses, then ϕ_1 and ϕ_2 are strongly mutually synchronous regularity preserving.*

PROOF. Suppose $A_1\phi_1$ and $A_2\phi_1$ are regular. Suppose also that (n, f_1, g_1, h_1) and (n, f_2, g_2, h_2) are sliding window inverses for ϕ_1 and ϕ_2 respectively, which are synchronised by $\psi : \mathbb{N} \to \mathbb{N}$.

Let $L \subseteq (X^+ \times X^+)$ be a synchronously regular binary relation over X. Then by Proposition 5.7, L is also a synchronously regular binary relation over X^1. Let

$$L' = \{(a_1\phi_1, a_2\phi_2) \mid (a_1, a_2) \ \in L \cap (A_1 \times A_2)\} \subseteq Y^+ \times Y^+.$$

Our objective is to show that L' is synchronously regular.

Certainly, we can write L' as a union

$$L' = L_1 \cup L_2 \cup L_3$$

where

$$L_1 = L' \cap (Y^+ \times Y^{\leq n+1}) = \bigcup_{v \in Y^{\leq n+1}} L' \cap (Y^+ \times \{v\}),$$

$$L_2 = L' \cap (Y^{\leq n+1} \times Y^+) = \bigcup_{u \in Y^{\leq n+1}} L' \cap (\{u\} \times Y^+)$$

and

$$L_3 = L' \cap (Y^{>n+1} \times Y^{>n+1}).$$

We shall show that each of L_1, L_2 and L_3 is synchronously regular, from which it will follow by Lemma 5.8(iv) that L' is synchronously regular.

First, we show that L_1 is synchronously regular. To this end, suppose $v \in Y^+$. Suppose first that $v \in A_2\phi_2$. Since ϕ_2 is injective, there is a unique path $b \in A_2$ such that $v = b\phi_2$. By Lemma 5.8 the language

$$K = \{a \mid (a,b) \in L\}$$

is regular. Now by Lemma 5.9 (the Sliding Window Lemma), the language $(K \cap A_1)\phi_1$ is regular, and so by Lemma 5.8(iii),

$$(K \cap A_1)\phi_1 \times \{v\}$$

is synchronously regular. But

$$(K \cap A_1)\phi_1 \times \{v\} = \{(a\phi_1, v) \mid (a,b) \in L, a \in A_1\} = L' \cap (Y^+ \times \{v\}),$$

so $L' \cap (Y^+ \times \{v\})$ is synchronously regular. Also, if $v \notin A_2\phi_2$, then $L' \cap (Y^+ \times \{v\})$ is empty, and hence synchronously regular.

But L_1 is a union of finitely many sets of this form (those with $v \in Y^{\leq n+1}$). It follows by Lemma 5.8(iv) that L_1 is synchronously regular. An entirely similar argument shows that L_2 is synchronously regular.

It remains only to show that L_3 is synchronously regular. Let

$$A = \{(a,b) \in A_1 \times A_2 \mid a\phi_1, b\phi_2 \in Y^{>n+1}\},$$

and define a new function

$$\phi : A\delta_{X^1} \to ((Y^1)^\$ \times (Y^1)^\$)^+$$

by

$$(a,b)\delta_{X^1}\phi = (a\phi_1, b\phi_2)\delta_{Y^1}.$$

Now we have

$$\begin{aligned}
L_3\delta_{Y^1} &= \{(a,b)\delta_{Y_1} \mid (a,b) \in L_3\} \\
&= \{(a,b)\delta_{Y_1} \mid (a,b) \in L', a, b \in Y^{>n+1}\} \\
&= \{(u\phi_1, v\phi_2)\delta_{Y_1} \mid (u,v) \in L \cap (A_1 \times A_2), u\phi_1, v\phi_2 \in Y^{>n+1}\} \\
&= \{(u\phi_1, v\phi_2)\delta_{Y_1} \mid (u,v) \in A \cap L\} \\
&= \{(u,v)\delta_{X_1}\phi \mid (u,v) \in A \cap L\} \\
&= (A \cap L)\delta_{X_1}\phi \\
&= (L\delta_{X_1} \cap A\delta_{X_1})\phi.
\end{aligned}$$

We shall show that ϕ is strongly regularity-preserving, by showing that it has a regular image and a sliding window inverse, and invoking Lemma 5.9 (the Sliding Window Lemma). It will then follow that $L_3\delta_{Y^1}$ is regular, and hence that L_3 is synchronously regular, as required.

First, we show that ϕ has a regular image, that is, that $A\delta_{X^1}\phi$ is a regular language over $(Y^1)^\$ \times (Y^1)^\$$. We have

$$\begin{aligned}
(A\delta_{X^1})\phi &= \{(a,b)\delta_{X^1}\phi \mid (a,b) \in A\} \\
&= \{(a\phi_1, b\phi_2)\delta_{Y^1} \mid (a,b) \in A\} \\
&= \{(a\phi_1, b\phi_2)\delta_{Y^1} \mid a \in A_1, b \in A_2, a\phi_1, b\phi_2 \in Y^{>n+1}\} \\
&= ((A_1\phi_1 \times A_2\phi_2) \cap (Y^{>n+1} \times Y^{>n+1}))\delta_{Y^1}
\end{aligned}$$

Now we know that $A_1\phi_1$, $A_2\phi_2$ and $Y^{>n+1}$ are regular, so it follows by Lemma 5.8(iii) and Lemma 5.8(iv) that

$$(A_1\phi_1 \times A_2\phi_2) \cap (Y^{>n+1} \times Y^{>n+1})$$

is synchronously regular, which by definition means that

$$((A_1\phi_1 \times A_2\phi_2) \cap (Y^{>n+1} \times Y^{>n+1}))\delta_{Y^1} = (A\delta_{X^1})\phi$$

is regular. Thus, ϕ has a regular image.

We shall now construct a sliding window inverse for ϕ. For $i = 1, 2$ we define functions

$$f_i' : \mathrm{Pref}_{n+1}(A\phi\pi_i) \to X^+,$$

$$f_i' : (y_1 \dots y_{n+1}) \mapsto (y_1 \dots y_n) f_i;$$

$$g_i' : \mathrm{Int}_{n+1}(A\phi\pi_i) \to X^+,$$

$$g_i' : (y_1 \dots y_{n+1}) \mapsto \begin{cases} \epsilon & \text{if } y_n = \$ \\ (y_1 \dots y_n) h_i & \text{if } y_n \neq \$, y_{n+1} = \$ \\ (y_1 \dots y_n) g_i & \text{otherwise} \end{cases};$$

and

$$h_i' : \mathrm{Suff}_{n+1}(A\phi\pi_i) \to X^+,$$

$$h_i' : (y_1 \dots y_{n+1}) \mapsto \begin{cases} \epsilon & \text{if } y_n = \$ \\ (y_1 \dots y_n) h_i & \text{if } y_n \neq \$, y_{n+1} = \$ \\ (y_1 \dots y_n) g_i (y_2 \dots y_{n+1}) h_i & \text{otherwise} \end{cases}.$$

We now define functions

$$f : \mathrm{Pref}_{n+1}(A\phi) \to ((X^1)^\$ \times (X^1)^\$)^+, \quad (x, y)\delta_{Y^1} \mapsto (xf_1', yf_2')\delta_{X^1},$$

$$g : \mathrm{Int}_{n+1}(A\phi) \to ((X^1)^\$ \times (X^1)^\$)^+, \quad (x, y)\delta_{Y^1} \mapsto (xg_1', yg_2')\delta_{X^1},$$

and

$$h : \mathrm{Suff}_{n+1}(A\phi) \to ((X^1)^\$ \times (X^1)^\$)^+, \quad (x, y)\delta_{Y^1} \mapsto (xh_1', yh_2')\delta_{X^1}.$$

We claim that $(n+1, f, g, h)$ is a sliding window inverse for ϕ.

First, let $x \in A_i$ with $|x\phi_i| > n + 1$ for $i = 1$ or $i = 2$. Suppose $x\phi_i = x_1 \dots x_m$ and let $k \geq m$ be an integer. If $k > m$ then define

$$x_{m+1} = x_{m+2} = \dots = x_k = \$.$$

We claim that

$$x = (x_1 \dots x_{n+1})f_i'(x_2 \dots x_{n+2})g_i'(x_3 \dots x_{n+3})g_i' \dots \\ \dots (x_{k-n-1} \dots x_{k-1})g_i'(x_{k-n} \dots x_k)h_i'. \tag{8}$$

To prove the claim, we shall consider separately the cases where $k = m$, $k = m+1$ and $k \geq m+2$.

First, suppose $k \geq m+2$. Certainly, by assumption, we have that

$$x = (x_1 \dots x_n)f_i(x_2 \dots x_{n+1})g_i(x_3 \dots x_{n+2})g_i \dots (x_{m-n+1} \dots x_m)h_i$$

Now by the definitions of f_i', g_i' and h_i', we have that

$$(x_1 \dots x_{n+1})f_i' = (x_1 \dots x_n)f_i,$$

$$(x_j \dots x_{j+n})g_i' = (x_j \dots x_{j+n-1})g_i \ \text{ for } 1 < j \leq m-n,$$

$$(x_{m-n+1} \dots x_{m+1})g_i' = (x_{m-n+1} \dots x_m)h_i,$$

$$(x_j \dots x_{j+n})g_i' = \epsilon \text{ for } m-n+1 < j < k-n$$

and

$$(x_{k-n} \dots x_k)h_i' = \epsilon.$$

It follows that (8) holds. A similar argument shows that (8) also holds when $k = m$ and when $k = m+1$.

Now suppose $w \in A\delta_{X^1}$, say $w = (w_1, w_2)\delta_{X^1}$. Then by the definition of A, we have $w_1\phi_1, w_2\phi_2 \in Y^{>n+1}$. Now by the definition of ϕ, we have $w\phi = (w_1\phi_1, w_2\phi_2)\delta_{Y^1}$. Suppose $w\phi = (x_1, y_1) \dots (x_k, y_k)$. Then from (8), we obtain

$$w_1 = (x_1 \dots x_{n+1})f_1'(x_2 \dots x_{n+2})g_1' \ (x_3 \dots x_{n+3})g_1' \dots \\ \dots (x_{k-n-1} \dots x_{k-1}) \ g_1'(x_{k-n} \dots x_k)h_1'$$

and similarly,

$$\begin{aligned}w_2 = (y_1 \dots y_{n+1}) f_2'(y_2 \dots y_{n+2}) g_2' \ (y_3 \dots y_{n+3}) g_2' \dots \\ \dots (y_{k-n-1} \dots y_{k-1}) g_2' \ (y_{k-n} \dots y_k) h_2'.\end{aligned}$$

Furthermore, by assumption, we have

$$\begin{aligned}|(x_1 \dots x_{n+1}) f_1'| &= |(x_1 \dots x_n) f_1| \\ &= 0\psi \\ &= |(y_1 \dots y_n) f_2| \\ &= |(y_1 \dots y_{n+1}) f_2'|\end{aligned}$$

from which it follows that

$$((x_1 \dots x_{n+1}) f_1', (y_1 \dots y_{n+1}) f_2') \delta_{X^1} = (x_1 \dots x_{n+1}, y_1 \dots y_{n+1}) \delta_{Y^1} f.$$

Now let m be the length of the shorter of $w_1\phi_1$ and $w_2\phi_2$. Then for every $1 < j \leq m - n$, we have

$$\begin{aligned}|(x_j \dots x_{j+n}) g_1'| &= |(x_j \dots x_{j+n-1}) g_1| \\ &= (j-1)\psi \\ &= |(y_j \dots y_{j+n-1}) g_2| \\ &= |(y_j \dots y_{j+n}) g_2'|,\end{aligned}$$

so that

$$((x_j \dots x_{j+n}) g_1', (y_j \dots y_{j+n}) g_2') \delta_{X^1} = (x_j \dots x_{j+n}, y_j \dots y_{j+n}) \delta_{Y^1} g.$$

Furthermore, either every $(x_j \dots x_{j+n}) g_1'$ for $j > m - n$ is the empty word or every $(y_j \dots y_{j+n}) g_2'$ for $j > m - n$ is the empty word.

It now follows from the definitions of f, g and h that

$$\begin{aligned} w &= (w_1, w_2)\delta_{X^1} \\ &= ((x_1, y_1) \dots (x_{n+1}, y_{n+1}))f((x_2, y_2) \dots (x_{n+2}, y_{n+2}))g \dots \\ &\qquad \dots ((x_{k-n}, y_{k-n}) \dots (x_k, y_k))h \end{aligned}$$

where $w\phi = (x_1, y_1)(x_2, y_2) \dots (x_k, y_k)$. This completes the proof that ϕ has a sliding window inverse.

We have shown that ϕ has a regular image and a sliding window inverse, and so by Lemma 5.9 (the Sliding Window Lemma), ϕ is strongly regularity preserving. Now we know that L is synchronously regular, which by definition means that $L\delta_{X^1}$ is regular. It follows that

$$L_3\delta_{Y^1} = (L\delta_{X^1} \cap A\delta_{X^1})\phi$$

is regular, and hence that L_3 is synchronously regular. Thus, L' is a union of synchronously regular binary relations, and hence is synchronously regular, as required. □

A special case of Lemma 5.10 gives a sufficient condition for a function to be strongly synchronous regularity preserving.

Corollary 5.11. *Let X and Y be graphs, $A \subseteq X^+$ and $\phi : A \to Y^+$ be an injective function. If $A\phi$ is regular and has a self-synchronised sliding window inverse, then ϕ is strongly synchronous regularity preserving.*

An application of Lemma 5.10 shows that synchronous regularity is preserved under simple, consistent changes to the ends of paths on one side of a binary relation.

Corollary 5.12. *Let X be a finite graph and $R \subseteq X^+ \times X^+$ be synchronously regular. Let $m \geq 0$ and let $f : X^m \to X^*$ be such that $xf\omega = x\omega$ for all*

$x \in X^m$. *Then the binary relation*

$$R' = \{(u, v(xf)) \mid (u, vx) \in R, u \in X^+, v \in X^*,\ x \in X^m\}$$

is synchronously regular.

PROOF. Let $x \in X^m$, and define $A_1 = X^+$ and $A_2 = X^*x$. Clearly, A_1 and A_2 are regular path languages. Let

$$R_x = R \cap (A_1 \times A_2).$$

By Lemma 5.8(iii) and Lemma 5.8(iv), R_x is synchronously regular. Now define functions

$$\phi_1 : A_1 \to X^+, w \mapsto w$$

and

$$\phi_2 : A_2 \to X^+, wx \mapsto w(xf).$$

Clearly, ϕ_1 and ϕ_2 are injective and the images $A_1\phi_1 = A_1 = X^+$ and $A_2\phi_2 = X^*(xf)$ are regular languages.

Let $n = |xf|$. Let $p : X^n \to X^*$ be given by $(x_1 \dots x_n)p = x_1$. Let $h_1 : X^n \to X^*$ be the inclusion map. Now $\mathrm{Suff}_n(A_2\phi_2) = \{xf\}$, so we can define $h_2 : \{xf\} \to X^*$ by $(xf)h_2 = x$.

Now for any $x_1 \dots x_r \in A_1$ we have

$$(x_1 \dots x_r)\phi_1 = x_1 \dots x_r$$

and then

$$(x_1 \dots x_n)p(x_2 \dots x_{n+1})p \dots (x_{r-n} \dots x_{r-1})p(x_{r-n+1} \dots x_r)h_1 = x_1 \dots x_r$$

so that (n, p, p, h_1) is a sliding window inverse for ϕ_1. Similarly, for any $x_1 \dots x_r \in A_2$ we have $x_{r-m+1} \dots x_r = x$ so that

$$(x_1 \dots x_r)\phi_2 = x_1 \dots x_{r-m}(xf) = x_1 \dots x_{r-m}y_1 \dots y_n$$

where $xf = y_1 \dots y_n$. Now we have

$$(x_1 \dots x_n)p(x_2 \dots x_{n+1})p \dots (x_{r-m}y_1 \dots y_{n-1})p(y_1 \dots y_n)h_2 = x_1 \dots x_{r-m}x = x_1 \dots x_r$$

so that (n, p, p, h_2) is a sliding window inverse for ϕ_2. Furthermore, the sliding window inverses are clearly synchronised by the constant function

$$\mathbb{N} \to \mathbb{N}, x \mapsto 1.$$

It follows by Lemma 5.10 (the Synchronous Sliding Windows Lemma) that ϕ_1 and ϕ_2 are mutually synchronous regularity preserving. Thus, the language

$$\{(u\phi_1, v\phi_2) \mid (u, v) \in R_x\}$$

is synchronously regular. But clearly we can write R' as a finite union

$$R' = \bigcup_{x \in X^m} \{(u\phi_1, v\phi_2) \mid (u, v) \in R_x\}$$

so by Lemma 5.8(iv), R' is synchronously regular as required. $\square$

CHAPTER 6

Automaticity

In this chapter, we present a concise but comprehensive survey of the existing theory of automatic and prefix-automatic semigroups. Where appropriate, we also discuss some new results, which will be proved directly, or arise as corollaries of results which are proved, in the chapters that follow.

We do not seek to define standard terminology which is used only in passing. The reader unfamiliar with terms used is advised to consult the articles referred to in the text, or the standard works mentioned in the introduction to Chapter 2.

6.1. Automatic Structures and Cross-sections

Let S be a finitely generated semigroup [monoid, group]. A *choice of (semigroup) generators* for S is a surjective semigroup morphism $\theta : A^+ \to S$ from a free semigroup A^+ onto S. If A is finite we say that θ is a *finite* choice of generators.

A *choice of representatives* for S is a triple (A, L, θ) where $\theta : A^+ \to S$ is a finite choice of semigroup generators for S, and $L \subseteq A^+$ is a language over A such that $L\theta = S$.

If θ maps L bijectively onto S, then (A, L, θ) is called a *cross-section* for S. If L is a regular language, then (A, L, θ) is called a *regular* choice of representatives for S. A regular choice of representatives which is also a cross-section is called a *regular cross-section* for S.

A *(right-right) automatic structure* for S is a choice of representatives (A, L, θ) for S, with the property that, for all $x \in A \cup \{\epsilon\}$, the binary relation

$$L_x = \{(a, b) \in L \times L \mid (ax)\theta = b\theta\}$$

is (right) synchronously regular, where ϵ denotes the empty word. Note that we have $(L_\epsilon)\pi_1 = L$, so by Lemma 5.8(ii), the definition implicitly requires that the language L be regular.

A semigroup S is called *(right-right) automatic* if there exists a right-right automatic structure for S. An automatic structure (A, L, θ) which is also a cross-section, that is, where L maps bijectively onto S, is called an *automatic cross-section*.

If (A, L, θ) is an automatic structure for S, then one can deduce that (A, K, θ) is an automatic cross-section for S, where $K \subseteq L$ is the language of all words in L which are minimal with respect to the *shortlex order* (see [**26**, Section 2.5]), amongst words in L representing the same element of S. It follows in particular that every automatic semigroup has an automatic cross-section [**14**, Corollary 5.6].

The notion of automaticity for *groups* was introduced in [**7**] and in [**26**]. The straightforward generalisation to semigroups was first made by Hudson [**43**], and has recently been explored in some detail by a number of authors [**13, 14, 15, 20, 21, 23, 33, 34, 36, 37, 38, 51, 52, 53, 54, 55, 56**]. In Chapter 8, we shall further generalise the concept of automaticity to semigroupoids and small categories.

6.2. Alternative Notions of Automaticity

When defining (right-right) automaticity, we made two arbitrary choices. We could require instead that the given binary relations be left synchronously regular (giving a notion of *left-right automaticity*), or that the binary relations of the form

$$\{(a, b) \in L \times L \mid (xa)\theta = b\theta\}$$

all $x \in A \cup \{\epsilon\}$ be left synchronously regular (*left-left automaticity*) or right synchronously regular (*right-left automaticity*).

The four possible definitions describe four classes of automatic semigroups, or fifteen classes once intersections are considered. In [**38**], it is shown that all fifteen are non-empty and distinct. Yet more classes of semigroups can be defined, by requiring that a semigroup have one choice of representatives which satisfies simultaneously a given combination of the above criteria. A particularly interesting class is that of semigroups with a choice of representatives which is simultaneously a right-right and a right-left automatic structure. Such semigroups are called *bi-automatic*.

In the case of groups, it is known that the classes of right-right, left-right, right-left and left-left automatic groups all coincide. However, it is an open question whether every (right-right) automatic group has a right-right automatic structure which is also a right-left automatic structure. In other words, it is not known whether every automatic group is bi-automatic. The property of bi-automaticity in groups is of particular interest, since it gives rise to a solvable conjugacy problem [**26**, Theorem 2.5.7].

We shall be predominantly concerned with right-right automatic structures and right-right automatic semigroups, which we shall refer to simply as *automatic structures* and *automatic semigroups* respectively.

For monoids, one can slightly weaken the definition of an automatic structure, by allowing an arbitrary choice of *monoid* generators, with the empty word representing the identity element. It is known that this alternative definition leads to the same class of automatic monoids, that is, that every monoid with an automatic structure with respect to a choice of monoid generators has also an automatic structure with respect to some choice of semigroup generators [**23**, Section 5]. We shall remark further on the relationship between these definitions in Section 6.4 below.

6.3. Prefix-Closure and Prefix-Automaticity

The *prefix-closure* of a choice of representatives (A, L, θ) is the choice of representatives $(A, \mathrm{Pref}(L), \theta)$. We say that (A, L, θ) is *prefix-closed* if it is its own prefix-closure, that is, if $\mathrm{Pref}(L) \subseteq L$. Prefix-closed automatic structures are particularly easy to reason about.

A *prefix-automatic structure* for a semigroup S is an automatic structure (A, L, θ) for S with the additional property that the binary relation

$$L'_{=} = \{(u, v) \in L \times \mathrm{Pref}(L) \mid u\theta = v\theta\}$$

is synchronously regular. A prefix-automatic structure which is also a cross-section is called a *prefix-automatic cross-section*. A semigroup is called *prefix-automatic* if it has a prefix-automatic structure. In fact (see Proposition 7.1 below), an automatic structure is prefix-automatic exactly if its prefix-closure is also an automatic structure. In particular, a semigroup is prefix-automatic if and only if it has a prefix-closed automatic structure.

Prefix-automatic (or "p-automatic") structures were introduced in the case of monoids of finite geometric type by Silva and Steinberg [**65, 66**].

In the group case, if (A, L, ρ) is an automatic structure, then $(A, \mathrm{Pref}(L), \rho)$ is a (prefix-closed) automatic structure [**26**, Theorem 2.5.9]. Thus, when proving results about automatic groups, one can often usefully assume the existence of a prefix-closed automatic structure. This result extends to right-cancellative semigroups – a result proved independently by Hoffmann [**33**, Proposition 9.3] in the general right-cancellative semigroup case, and by Silva and Steinberg [**66**, Theorem 8.1] in the case of right-cancellative monoids – but not to semigroups in general. For example, Hoffmann [**33**, Example 9.1] shows that the semigroup with presentation

$$\langle a, b, x \mid ax = bx, ab = ba \rangle$$

has an automatic structure (indeed, an automatic cross-section) the prefix-closure of which is not an automatic structure. Of course, this leaves open the possibility that the given semigroup has a different prefix-closed automatic structure with respect either to the same, or to a different, choice of generators. Indeed, the following two questions remain open.

QUESTION 6.1. *Is every automatic semigroup prefix-automatic?*

QUESTION 6.2. *Does every semigroup which has an automatic structure with respect to a given choice of generators have a prefix-automatic structure with respect to the same choice of generators?*

By the results quoted above, both questions have positive answers in the cases of groups and of right-cancellative semigroups. We shall see later (Corollary 7.10) that the first question has a positive answer in the case of completely simple and completely 0-simple semigroups.

We have seen that every automatic semigroup has an automatic cross-section, and that every automatic group has a prefix-closed automatic structure. It remains an open question whether every prefix-automatic semigroup (or even every automatic group [**26**, Example 2.5.10]) has an automatic structure enjoying both of these properties at once, that is, a prefix-closed automatic cross-section.

QUESTION 6.3. *Does every prefix-automatic semigroup (or every automatic group) have a prefix-closed automatic cross-section?*

However, one can show (see [**65**] and also Proposition 7.1 and Corollary 8.7 below) that every prefix-automatic semigroup has a prefix-automatic cross-section. Thus, when working with a prefix-automatic semigroup, we can enjoy many of the benefits of a prefix-closed automatic structure and an automatic cross-section at once.

6.4. Change of Generators

In the group case, it is known [**26**, Theorem 2.3.5] that if a group G has an automatic structure (A, L, θ) and $\sigma : B^+ \to G$ is an alternative finite choice of (semigroup) generators for G, then G has an automatic structure (B, K, σ) for some $K \subseteq B^+$. Thus, in the group case, automaticity is in some sense an invariant property of groups, independent of the choice of generators.

The proof of this result employs geometric arguments, which do not easily translate into a more general semigroup setting (see Section 6.6 below). A geometric proof does extend the result to the case of completely simple semigroups [**14**, Proposition 4.3], and an alternative proof method has been used to show that the result extends to monoids (again, considered with semigroup generators) [**23**, Theorem 1.1]. However, it is known not to hold for semigroups in general.

For example, in [**14**, Example 4.5], it is shown that the subsemigroup of the free semigroup $\{a, b, c\}^+$ generated by the elements c, ac, ca, ab and $baba$ has no automatic structure with respect to this choice of generators, but does have an automatic structure if one introduces an additional redundant generator $abab$.

Nor does invariance under change of generators hold for monoids if the definition of an automatic structure is weakened to allow an arbitrary choice of *monoid* generators, with the empty word representing the identity element [**23**, Section 5]. This apparent discrepancy arises from the use of *synchronous* regularity in the definition of automaticity, and can be explained intuitively as follows. When a pair of words is read by a synchronous automaton, the relative *speed* at which the words are read may determine whether the automaton needs to store an unbounded amount of information, in order to achieve its task of recognising a given binary relation. When considering semigroup generators for a monoid M, one has a degree of flexibility to choose the speed at which a given word is read, since any word can be "padded" by the insertion of non-empty words representing the identity. This flexibility may be lost if no non-empty words in the generators represent an identity element.

Silva and Steinberg [**65**, Theorem 6.3] use a similar proof to that of [**23**, Theorem 1.1] to show that prefix-automaticity in monoids (considered with semigroup generators) is also invariant under change of generators.

6.5. Closure Properties

Subsemigroups and Extensions. Let (A, L, σ) be a regular choice of representatives for a semigroup S. A subsemigroup T of S is called *L-regular*, if $L \cap T\sigma^{-1}$ is regular. In the group case, if (A, L, σ) is an automatic structure for a group G then every L-regular subgroup of G is automatic [**26**, Theorem 8.3.4]. In particular, one can show that every subgroup of finite index in G is L-regular, from which it follows that every subgroup of finite index in an automatic group is automatic [**26**, Example 8.3.2].

In the semigroup case, there is no such general result. Indeed, as Silva and Steinberg [**66**, Section 5] observe, an L-regular sub*monoid* of a finitely generated free *group* need not even be finitely generated. Rather, there are a number of results giving sufficient conditions for automaticity in subsemigroups of automatic semigroups.

Let T be a cofinite (or "finite Rees index") subsemigroup of a semigroup S. Then S is automatic if and only if T is automatic [**33, 36**]. This generalises the earlier results that the class of automatic semigroups is closed under the adjoining of and, where applicable, removal of, an identity or zero element [**14**, Theorem 7.2 & Proposition 3.13].

Hoffmann, in his proof of [**33**, Theorem 5.1], shows that if the subsemigroup T is prefix-automatic then the containing group S is prefix-automatic. The proof is there misdescribed as a proof of the converse statement, that is, that if S is prefix-automatic then T is prefix-automatic. We shall see later (Theorem 7.3) that the converse does indeed hold. We note that Silva and Steinberg [**66**, Theorem 6.1 & Proposition 6.2] independently prove equivalent results in the case that S and T are monoids with T a cofinite ideal of S. (Note that, although a proper ideal of a monoid cannot be a submonoid, it can be a monoid in its own right, with a different identity element.)

Duncan *et al* [**23**, Lemma 4.2] give a useful, if rather technical, sufficient condition for automaticity in a subsemigroup which is a retract of an automatic semigroup. Silva and Steinberg [**66**, Lemma 5.2] give an analogous condition for prefix-automatic monoids, along with a number of other useful sufficient conditions under which prefix-automaticity in a submonoid implies prefix-automaticity in the containing monoid, or vice versa.

Rees Quotients. If M is a monoid with an automatic [prefix-automatic] structure (A, L, ρ) and I is an L-regular ideal of M, then the Rees quotient of M by I is automatic [respectively, prefix-automatic] [**66**, Theorem 6.3]. The following question naturally arises.

QUESTION 6.4. *Does the above statement hold for M a semigroup without identity?*

Direct, Semidirect and Wreath Products. A direct product of two automatic monoids is always automatic [**14**, Theorem 6.4]. Similarly, a direct product of prefix-automatic monoids is always prefix-automatic [**66**, Theorem 8.3]. A direct product of two automatic semigroups need not even be finitely generated, but is automatic whenever it is finitely generated [**13**]. One is drawn to ask whether the same holds for prefix-automatic semigroups.

QUESTION 6.5. *Is a finitely generated direct product of two prefix-automatic semigroups necessarily prefix-automatic?*

It is not yet known if there is a converse for these results, even in the case of groups.

QUESTION 6.6. *If S and T are semigroups (or monoids, or groups) such that the direct product $S \times T$ is automatic [prefix-automatic], are S and T necessarily automatic [respectively, prefix-automatic]?*

It is known that if S is an automatic semigroup and T is a finite monoid acting on S on the left by endomorphisms, then the semidirect product of S with T is automatic [**36**, Corollary 5.2]. An immediate corollary is that a wreath product of an automatic semigroup with a finite monoid is automatic whenever it is finitely generated [**36**, Corollary 5.3, Remark 5.4]. Both these results extend to prefix-automatic monoids [**66**, Section 8].

Free and Amalgamated Products. Let H_1 and H_2 be groups, and G be the amalgamated free product of H_1 and H_2 over some common finite subgroup. It is known that G is automatic [bi-automatic] if [**7**] and only if [**26**, Theorem 12.1.4] H_1 and H_2 are automatic [respectively, bi-automatic]. In particular, this result applies when the common subgroup is trivial, so that G is the free product of H_1 and H_2.

Let S and T be semigroups. Then the *semigroup* free product of S and T is automatic if and only if both S and T are automatic. [**14**, Theorem 6.1]. Similarly, if M and N are monoids, then the *monoid* free product of M and N is automatic if [**14**, Theorem 6.2] and only if [**23**, Theorem 1.2] M and N are automatic. An analogous result holds for prefix-automaticity in monoids with monoid free products [**66**, Theorems 8.1 & 8.2], and we shall see later (Corollary 7.5) that it holds also for prefix-automaticity in general semigroups with semigroup free products.

Hoffmann *et al* show that any finitely generated ideal of a free product of automatic semigroups is automatic [**36**, Corollary 5.1].

In the group case, various other conditions are known [**7**] under which an amalgamated free product of automatic groups over an automatic subgroup is automatic. Both the conditions and the methods of proof seem to be particular to groups, so it seems unlikely that these results will extend naturally to monoids or semigroups. However, one can still ask the following rather general question.

QUESTION 6.7. *Under what conditions is an amalgamated product of automatic or prefix-automatic semigroups automatic or prefix-automatic?*

Rees Matrix Semigroups. A Rees matrix semigroup (with or without zero) over a group G, with a finite sandwich matrix, is automatic if and only if G is automatic [**15**, Theorem 1.1]. We saw in Chapter 2 that every completely simple or completely 0-simple semigroup arises as a Rees matrix semigroup (with or without zero) over a group. Thus, this result characterises automaticity in completely simple and completely 0-simple semigroups. However, it is known that a semilattice of automatic groups need not be automatic [**15**, Example 5.1].

Let M be a Rees matrix semigroup over a semigroup S. Various results are known about the relationship between the properties of automaticity and prefix-automaticity in M and S [**66, 21**]. All of these arise by specialising to the semigroup case results which we shall prove for semigroupoids (see Chapters 7 and 9), so we shall not discuss them in detail here.

Reinterpreting Presentations. If S is an automatic semigroup with presentation $\langle A \mid R\rangle$, then the group G defined by the presentation $\langle A \mid R\rangle$ need not be automatic [**15**, Section 5].

Conversely, if G is an automatic group with presentation $\langle A \mid R\rangle$ then the semigroup defined by the same presentation need not be automatic. Indeed, if $\langle A \mid R\rangle$ is any presentation for a semigroup with zero, then the group with presentation $\langle A \mid R\rangle$ is necessarily trivial and hence automatic. Thus, any non-automatic semigroup with zero shares a presentation with an automatic group, and thus provides a counterexample. This observation is due to N. Ruškuc.

A related question concerns the special case in which S embeds naturally in G.

QUESTION 6.8. *Let S be a semigroup with presentation $\langle A|R\rangle$, and G the group defined by the same presentation. Suppose S is the positive semigroup of G. Is automaticity in G a necessary and/or sufficient condition for automaticity (or prefix-automaticity) in S?*

6.6. Automaticity and Geometry

We remarked in Chapter 1 that combinatorial group theory is intimately connected with geometric group theory, but that it is often difficult to extend the geometric connection to the combinatorial study of semigroups. The theory of automaticity is no exception to this general rule.

It is well-known that automatic structures for groups are characterised by a geometric condition on the associated Cayley graphs, known as the *fellow-traveller property*. For a formal definition and proof, see [**26**, Theorem 2.3.5]. We shall content ourselves with an informal explanation. Intuitively, a regular choice of representatives (A, L, ρ) for a group G satisfies the fellow traveller property if paths in the Cayley graph of G with respect to A, which are labelled by words in L and which eventually converge to within a distance of 1, never diverge beyond some given distance.

If we seek to apply this condition to semigroups, we must first decide what we mean by "distance" in a semigroup Cayley graph. One could take the distance between two vertices to be the shortest length of an undirected path connecting the vertices. This gives a metric on the Cayley graph (or at least, on its connected components), but in dismissing the directional information in the graph, we obtain a geometric model which fails to capture fully much of the combinatorial structure of the semigroup. The resulting *undirected fellow traveller property* turns out to be necessary but not sufficient for automaticity in general semigroups [**14**]. This condition is rather more meaningful in semigroups which admit of some notion of an inverse. For example, the undirected fellow traveller property is both necessary and sufficient for automaticity in completely simple semigroups [**15**, Proposition 3.12]. (This closely parallels the situation for *word hyperbolic semigroups* [**22**], where a geometric characterisation akin to that in groups has so far been obtained only for completely simple semigroups [**28**].)

Alternatively, one might define the distance between two vertices to be the shortest length of a *directed* path connecting them, *in either direction*. However, we still discard structural information (namely, the direction of the path). Moreover, "distance" now fails to satisfy the triangle inequality, and so does not define a metric. The resulting *directed fellow traveller property* is neither necessary nor sufficient for automaticity in general semigroups [**14**], although it is sufficient in the case of left-cancellative semigroups [**33**].

Finally, we could define the distance from a vertex a to a vertex b to be the shortest length of a directed path from a to b. Here, we capture much more information about the combinatorial structure of the semigroup, but at a price – this notion of "distance" is not symmetric! The resulting version of the fellow traveller property, which we shall call the *strongly directed fellow traveller property* is clearly stronger than the previous one, and hence cannot be a necessary condition for automaticity. Whether this condition is sufficient for automaticity does not seem to have been investigated.

QUESTION 6.9. *Is the strongly directed fellow traveller property sufficient to guarantee automaticity in general semigroups?*

More success in seeking a geometric characterisation can be achieved by restricting attention to classes of semigroups whose Cayley graphs satisfy certain properties. In the case of monoids of finite geometric type, [**65**, Theorem 7.6] gives a (largely) geometric condition equivalent to prefix-automaticity. An alternative approach is taken in [**33**, Section 8.3], where automatic structures (A, L, ρ) are characterised in terms of the number of "continuation graphs" associated with an automaton accepting the language L. This characterisation is certainly of interest, although the degree to which it is "geometric" is debatable.

6.7. Examples and Counterexamples

Notable examples of automatic semigroups include

- finite semigroups [**14**, Example 4.1];
- finitely generated free semigroups [**14**, Example 4.3];
- finitely generated subsemigroups of free semigroups [**14**, Theorem 8.1];
- the bicyclic monoid [**14**, Example 4.2];
- finitely generated subsemigroups of the bicyclic monoid [**20**, Theorem 8.2]; and
- the fundamental four spiral semigroup [**21**, Example 3.4].

Proving a given finitely generated semigroup S *not* to be automatic is, in general, rather difficult. The dependence on choice of generators is not the obstacle it may first seem, since it can be bypassed, by considering instead automaticity of the monoid S^1. However, one is still left with the task of demonstrating the non-existence of a language L with certain properties.

An automatic structure implies the existence of an automatic cross-section, which is in particular a regular cross-section. Some semigroups can be shown not to have a regular cross-section, from which one concludes that they lack also an automatic structure. This approach is used in [**19**] to show that finitely generated free inverse semigroups are not automatic. However, this method is far from universal. For example, it is known [**37**] that not every commutative semigroup is automatic, but the present author has shown that every commutative semigroup does have a regular cross-section.

Notable examples of finitely generated semigroups known *not* to be automatic include

- free inverse semigroups and monoids [**19**]; and
- infinite periodic monoids [**65**, Theorem 10.3].

6.8. Commutative Semigroups and Automaticity

It is known [**26**, Theorem 4.3.1] that all finitely generated abelian groups are automatic, and indeed shortlex automatic (see [**26**, Section 2.5]). This property seems to emanate not from the axiom of commutativity as such, but rather from the simple structure which such groups enjoy. In particular, every finitely generated abelian group is a direct product of finitely many cyclic groups.

The natural question of whether every finitely generated commutative semigroup is automatic was posed by Campbell *et al* [**14**, Question 4.6] and answered in the negative by Hoffmann [**37**]. He showed that the commutative semigroup with presentation

$$\langle a, b, x, y \mid aax = bx, bby = ay, ab = ba, ax = xa, ay = ya,$$
$$bx = xb, by = yb, xy = yx\rangle$$

has no automatic structure. It remains an open question whether there is a simple characterisation of automaticity or prefix-automaticity in commutative semigroups. Two conditions have been conjectured as potentially sufficient. First, the above counterexample is not of finite geometric type. The following question was asked in [**37**] (where the term *bounded indegree* is used in place of "finite geometric type") and in [**65**, Section 10].

QUESTION 6.10. *Is every finitely generated commutative semigroup of finite geometric type, automatic, or indeed prefix-automatic?*

Secondly, in [**37**] it is observed that the counterexample given above is such that Green's $\mathscr{H}$ binary relation (see [**40**]) has infinitely many equivalence classes. The following question is posed.

QUESTION 6.11. *Is every finitely generated commutative semigroup with finitely many $\mathscr{H}$-classes automatic?*

Note that Green's binary relations $\mathscr{H}$, $\mathscr{R}$ and $\mathscr{L}$ coincide in a commutative semigroup.

6.9. Presentation Properties

We have already observed that automatic semigroups are, by definition, finitely generated. In the case of automatic groups, it is known that automaticity is a sufficient condition for the existence of a finite presenation [**26**, Theorem 2.3.12]. This result extends to automatic completely simple semigroups [**15**, Corollary 1.2], but not to automatic semigroups in general. For example, Campbell *et al* [**14**, Example 4.4] show that the semigroup with (infinite) presentation

$$\langle a, b \mid ab^i a = aba \quad (i \geq 2)\rangle$$

has an automatic structure but no finite presentation. In particular, they show that the language

$$L = \{b\}^*\{a, ab\}^*\{b\}^* \setminus \{\epsilon\}$$

gives an automatic structure. We observe that this language is prefix-closed, demonstrating that even prefix-automatic semigroups need not be finitely presentable.

6.10. Complexity Issues and Rewriting Systems

A number of interesting results concern issues of computational complexity in automatic semigroups, or relate automatic structures to certain kinds of string-rewriting systems. A detailed discussion of these topics would be beyond the scope of this work but, for completeness, we summarise the key results in this area. We refer the reader to [**8**] for an introduction to string-rewriting systems, and to the sources cited below for precise definitions of particular terms.

It is known that the word problem for any automatic semigroup is solvable in quadratic time [**14**, Corollary 3.7]. Given an automatic structure for a monoid, it is decidable whether the monoid is right-cancellative, and whether it is a group [**65**, Proposition 9.4].

Every automatic group has a Dehn function of at most quadratic complexity [**26**, Theorem 2.3.12]. By contrast, one can construct automatic (indeed, bi-automatic) monoids with Dehn functions of arbitrarily great complexity [**52**].

Not every group (and hence not every semigroup) with a finite convergent presentation is automatic [**29**]. It is not known whether every automatic group has a finite convergent presentation, but it is known that there are automatic monoids without finite convergent presentations [**55**].

In [**51**] it is shown that a monoid has a prefix-closed automatic cross-section if and only if it has a presentation which constitutes an s-regular canonical prefix-rewriting system. This result may offer some insight into Question 6.3 above. It is also known [**54**] that a monoid with a presentation which constitutes a finite, special [respectively, a regular, monadic], confluent string-rewriting system always has a choice of representatives which is simultaneously a left-left, left-right, right-left and right-right automatic structure [respectively, a left-left and right-right automatic structure].

6.11. Related Ideas

Certain other classes of semigroups are defined by the existence of a choice of representatives with certain formal language theoretic properties. These include the classes of *asynchronous automatic semigroups* [**34**], *word hyperbolic semigroups* [**22, 28**], and *rational semigroups* [**57, 64**]. The first two classes, as with automatic semigroups, are generalisations of well-known classes of groups [**26, 30**]. The latter class is of a rather different nature, having arisen from an attempt to characterise the monoids in which Kleene's Theorem holds (see [**64**]). A detailed study of the relationships between these classes and automatic semigroups can be found in [**34**].

CHAPTER 7

Automatic Semigroups

In this chapter, we introduce some new results about automatic semigroups and monoids. Some of these are proved directly, while others arise as corollaries of results about semigroupoids and categories which we shall prove in Chapters 8 and 9.

It seems most appropriate to group together all our results concerning automatic semigroups, so that the reader is given a full picture of the implications for semigroup theory. Those who prefer a strictly logical development might prefer first to read Chapter 8 up to and including the proof of Corollary 8.7, then Sections 7.1 and Section 7.2, followed by the remainder of Chapter 8 and finally the remainder of Chapter 7.

7.1. Automaticity and Prefix-Automaticity

The following proposition, which slightly generalises [**65**, Corollary 5.4 and Proposition 5.8], is a simple specialisation of Proposition 8.5 and Corollary 8.7.

PROPOSITION 7.1. *Let S be a semigroup. Then the following are equivalent:*

(i) *S is prefix-automatic.*
(ii) *S has a prefix-closed automatic structure.*
(iii) *S has a prefix-closed prefix-automatic structure.*
(iv) *S has a prefix-automatic cross-section.*

7.2. Prefix-Automaticity and Cofinite Subsemigroups

In this section, we show that every cofinite subsemigroup of a prefix-automatic semigroup is prefix-automatic. We observe that in [**33**] a proof is mistakenly claimed of the (by Proposition 7.1, equivalent) statement that every cofinite subsemigroup of a semigroup with a prefix-closed automatic structure has a prefix-closed automatic structure. In fact, the proof given there is a proof of the converse statement. The same proof appears, correctly described, in [**36**].

LEMMA 7.2. *Let S be a semigroup with a prefix-automatic structure (A, L, ρ). Let T be a cofinite semigroup of S. Then the language* $\mathrm{Pref}(L) \cap T\rho^{-1}$ *is regular.*

PROOF. For each $s \in S \setminus T$, let w_s be some representative in L for s. Then by the definition of $L'_=$ and Lemma 5.8(iv) and (ii), we see that

$$\mathrm{Pref}(L) \cap s\rho^{-1} = (L'_= \cap (\{w_s\} \times \mathrm{Pref}(L))\pi_2$$

is regular. But then

$$\mathrm{Pref}(L) \cap T\rho^{-1} = \mathrm{Pref}(L) \setminus (\bigcup_{s \in S \setminus T} \mathrm{Pref}(L) \cap s\rho^{-1})$$

is regular, as required. □

We are now ready to state and prove the main result of this section.

THEOREM 7.3. *Let S be a semigroup and T a cofinite subsemigroup of S. If S is prefix-automatic, then T is prefix-automatic.*

PROOF. Suppose S is prefix-automatic. By Proposition 7.1, we can choose a prefix-automatic cross-section (A, L, ρ) for S.

By [**36**, Lemma 4.1], there exists an integer k such that for every

$$w \in \mathrm{Fact}(L) \cap A^{\geq k} \cap A^{<2k},$$

we have $w\rho \in T$. We define a new alphabet B by

$$B = \{b_w \mid w \in \mathrm{Fact}(L) \cap A^{<2k} \cap T\rho^{-1}\}$$

and a new morphism $\sigma : B^+ \to T$ by $b_w\sigma = w\rho$.

We define a language

$$V = T\rho^{-1} \cap \mathrm{Pref}(L) = \{w \in \mathrm{Pref}(L) \mid w\rho \in T\}.$$

By Lemma 7.2, V is regular. We now define a function $\phi : V \to B^+$ by

$$w\phi = \begin{cases} b_w & \text{if } w \in A^{\leq k} \\ b_{w_1}b_{w_2}\dots b_{w_n} & \text{if } w = w_1w_2\dots w_n \text{ with } w_1,\dots,w_{n-1} \in A^k \\ & \qquad\qquad \text{and } w_n \in A^{\geq k} \cap A^{<2k}. \end{cases}$$

Let $K = (L \cap V)\phi$. Clearly, ϕ is injective and $w\phi\sigma = w\rho$ for all $w \in V$. Also, since (A, L, ρ) is a cross-section, we know that the restriction of ρ to L is a bijection. It follows that if $v, w \in K$ are such that $v\sigma = w\sigma$, then $v = x\phi$ and $w = y\phi$ for some $x, y \in L \cap V$ so that

$$x\rho = x\phi\sigma = v\sigma = w\sigma = y\phi\sigma = y\rho$$

so that $x = y$ and $v = x\phi = y\phi = w$. Thus, the restriction of σ to K is injective. Furthermore, if $t \in T$ then $t = x\rho$ for some $x \in L$, and certainly $x \in T\rho^{-1}$ so that $x \in V$. Now $x\phi \in (L\cap V)\phi = K$ and $(x\phi)\sigma = x\rho = t$. Thus, the restriction of σ to K is surjective.

We have shown that (B, K, σ) is a cross-section for T. We claim that (B, K, σ) is, in fact, a prefix-automatic cross-section for T. We begin by showing that ϕ has a regular image and a self-synchronised sliding window inverse.

We show first that the image $V\phi$ is regular. Define a new function

$$f : B^+ \to A^+$$

by

$$(b_{w_1} \dots b_{w_n})f = w_1 \dots w_n.$$

Clearly f is a morphism, and regular languages are closed under inverse morphisms. We have seen that V is regular, so it follows that the language

$$Vf^{-1} = \{b_{w_1} \dots b_{w_n} \mid w_1 \dots w_n \in V\}$$

is regular. But now we see that

$$V\phi = Vf^{-1} \cap (B_1 \cup B_2^* B_3)$$

where

$$B_1 = \{b_w \mid w \in A^{\leq k}\},$$

$$B_2 = \{b_w \mid w \in A^k\}$$

and

$$B_3 = \{b_w \mid w \in A^{\geq k} \cap A^{<2k}\}$$

are all finite. Thus, $V\phi$ is regular.

Now with f defined as in the previous paragraph, we claim that $(1, f, f, f)$ is a sliding window inverse for ϕ. To see this, suppose $w \in V$, $n > 1$ and $y_1, \dots, y_n \in B$ are such that $y_1 \dots y_n = w\phi$. Since $n > 1$, it follows from the definition of ϕ that we must have $w = w_1 \dots w_n$ for some $w_1, \dots, w_{n-1} \in A^k$ and $w_n \in A^{\geq k} \cap A^{<2k}$, and that

$$y_1 \dots y_n = w\phi = b_{w_1} \dots b_{w_n}.$$

But now we have

$$\begin{aligned}(y_1)f(y_2)f(y_3)f \dots (y_{n-1})f(y_n)f &= (b_{w_1})f(b_{w_2})f \dots (b_{w_{n-1}})f(b_{w_n})f \\ &= w_1 w_2 \dots w_n \\ &= w\end{aligned}$$

as required. We have already seen that ϕ has a regular image, so by Lemma 5.9 (the Sliding Window Lemma), it follows that ϕ is strongly regularity preserving.

Furthermore, if we define $\psi : \mathbb{N} \to \mathbb{N}$ by $n\psi = k$ for all $n \in \mathbb{N}$, we see that $(1, f, f, f)$ is self-synchronised by ψ. We observed above that ϕ is injective, so it follows by Corollary 5.11 that ϕ is strongly synchronous regularity preserving.

Now since L is regular and ϕ is strongly regularity preserving, we deduce that $K = (L \cap V)\phi$ is regular. Since (B, K, σ) is a cross-section for T, it follows by Lemma 5.8(v) that

$$K_= = \{(w, w) \mid w \in K\}$$

is synchronously regular.

Now let $b \in B$. Then $b = b_w$ for some w, and by Proposition 8.2, L_w is synchronously regular. But we have

$$\begin{aligned} K_b &= \{(u, v) \in K \times K \mid (ub)\sigma = v\sigma\} \\ &= \{(x\phi, y\phi) \mid x, y \in L \cap V, ((x\phi)b)\sigma = y\phi\sigma\} \\ &= \{(x\phi, y\phi) \mid x, y \in L \cap V, (xw)\rho = y\rho\} \\ &= \{(x\phi, y\phi) \mid (x, y) \in L_w \cap (V \times V)\}. \end{aligned}$$

It follows by Lemma 5.8(iii) and Lemma 5.8(iv) that $L_w \cap (V \times V)$ is synchronously regular. Since ϕ is strongly synchronous regularity preserving, it follows that K_b is synchronously regular.

We have shown that (B, K, σ) is an automatic cross-section for L. It remains only to show that it is a prefix-automatic structure, that is, that the binary relation

$$K'_= = \{(u, v) \in K \times \mathrm{Pref}(K) \mid u\sigma = v\sigma\}$$

is synchronously regular.

We claim that every non-empty word in $\mathrm{Pref}(K)$ can be written as $x\phi$ for some $x \in V$. By definition, every word in K itself can certainly be written as $x\phi$ for some $x \in L \cap V \subseteq V$. Now consider a non-empty word $v \in \mathrm{Pref}(K) \setminus K$. Then there is a non-empty word $v' \in B^+$ such that $vv' \in K$. Now $K = (L \cap V)\phi$, so we must have $vv' = w\phi$ for some word $w \in L \cap V$. Since v and v' are both non-empty, vv' has length at least 2. Thus, from the definition of ϕ, we must have $w = w_1 w_2 \ldots w_n$ for some $n \geq 2$, $w_1, \ldots, w_{n-1} \in A^k$ and $w_n \in A^{\geq k} \cap A^{<2k}$. But now we have

$$vv' = w\phi = b_{w_1} b_{w_2} \ldots b_{w_n}$$

so in particular we have

$$v = b_{w_1} \ldots b_{w_j} = (w_1 \ldots w_j)\phi$$

for some j with $1 \leq j < n$. Let $x = w_1 \ldots w_j$. Then $x \in \mathrm{Pref}(L)$ (since it is a prefix of $w \in L$). Also, by the original assumption on k, we have $w_1\rho, w_2\rho, \ldots, w_j\rho \in T$, so that

$$x\rho = (w_1\rho)(w_2\rho) \ldots (w_j\rho) \in T$$

and $x \in T\rho^{-1}$. This completes the proof that every word in $\mathrm{Pref}(K)$ is of the form $x\phi$ for some $x \in V$.

Now, noting that $V \subseteq \mathrm{Pref}(L)$, we have

$$\begin{aligned}
K'_{=} &= \{(u, v) \mid u \in K, v \in \mathrm{Pref}(K), u\sigma = v\sigma\} \\
&= \{(x\phi, y\phi) \mid x \in L \cap V, y \in V, y\phi \in \mathrm{Pref}(K), x\phi\sigma = y\phi\sigma\} \\
&= \{(x\phi, y\phi) \mid x \in L \cap V, y \in V, y\phi \in \mathrm{Pref}(K), x\rho = y\rho\} \\
&= \{(x\phi, y\phi) \mid (x, y) \in L'_{=} \cap (V \times V), y\phi \in \mathrm{Pref}(K)\} \\
&= \{(x\phi, y\phi) \mid (x, y) \in L'_{=} \cap (V \times V)\} \cap (B^+ \times \mathrm{Pref}(K)).
\end{aligned}$$

Now B^+ is certainly regular, and K is regular so $\mathrm{Pref}(K)$ must be regular. By Lemma 5.8(iii), it follows that $B^+ \times \mathrm{Pref}(K)$ is synchronously regular. We know that $L'_=$ is synchronously regular and ϕ is strongly synchronous regularity preserving, so

$$\{(x\phi, y\phi) \mid (x, y) \in L'_= \cap (V \times V)\}$$

is synchronously regular. Now by Lemma 5.8(iv), it follows that $K'_=$ is synchronously regular, as required.

Thus, (B, K, σ) is a prefix-automatic structure (indeed, a prefix-automatic cross-section) for S, as required. □

Combining Theorem 7.3 with Proposition 7.1 and [**36**, Theorem 1.1 and remarks preceding proof], we obtain the following result.

THEOREM 7.4. *Let S be a semigroup and T a cofinite subsemigroup of S. Then S is prefix-automatic if and only if T is prefix-automatic.*

We shall rely on this result in Chapter 8, when we generalise it to the case of semigroupoids (Theorem 8.8). We note that, while our proof here relies on Proposition 8.2, and (via Proposition 7.1) on Proposition 8.5 and Corollary 8.7, there is no dependence on any later material in Chapter 8. Thus, there is no circular dependence in our logic.

An immediate corollary of Theorem 7.4 is that some results of [**66**] extend to give a characterisation of prefix-automaticity in free products of semigroups.

COROLLARY 7.5. *Let S and T be semigroups. Then S and T are prefix-automatic if and only if their semigroup free product is prefix-automatic.*

PROOF. Let $S * T$ denote the semigroup free product of S and T. Let $\overline{S}$ and $\overline{T}$ be the monoids obtained by adjoining a new identity element to S and T respectively, and let $\overline{S} *_1 \overline{T}$ denote the monoid free product (amalgamating identities) of $\overline{S}$ and $\overline{T}$. Then it is easily verified that $\overline{S} *_1 \overline{T}$ is isomorphic to $S * T$ with an identity adjoined.

Suppose first that S and T are prefix-automatic. Then by Theorem 7.4, $\overline{S}$ and $\overline{T}$ are prefix-automatic. By [**66**, Theorem 8.1], $\overline{S} *_1 \overline{T}$ is prefix-automatic. Now by Theorem 7.4, $S * T$ is prefix-automatic.

Conversely, if $S * T$ is prefix-automatic then we deduce by Theorem 7.4 that $\overline{S} *_1 \overline{T}$ is prefix-automatic, by [**66**, Theorem 8.2] that $\overline{S}$ and $\overline{T}$ are prefix-automatic, and then by Theorem 7.4 that S and T are prefix-automatic. $\square$

7.3. Rees Matrix Semigroups over Semigroups

In this section, we discuss new results for the theory of automatic and prefix-automatic semigroups, which can be obtained by specialising the results to be proved in the following chapters to semigroups (that is, single-object semigroupoids), monoids (single-object small categories) and groups (single-object groupoids).

The weak right-ideal-generation condition, introduced in Chapter 9, simplifies in the semigroup case:

Proposition 7.6. *Let S be a semigroup, 0 a symbol not in S and P a $\Lambda \times I$ matrix with entries drawn from $S \cup \{0\}$. Then the non-identity elements of S are weakly right-ideal-generated by a row cross-section of P if and only if there exists $\lambda \in \Lambda$ such that every non-identity element of S can be written in the form $P_{\lambda i}$ or $P_{\lambda i} s$ for some $i \in I$ and (where appropriate) $s \in S$.*

We remark that this condition is weaker than the requirement that S be the principal right ideal generated by some single entry in the matrix P, which was shown in [**21**] to be sufficient for automaticity in a Rees matrix semigroup to imply automaticity in the underlying semigroup. It is certainly also weaker than the requirement that S be a monoid, and the sandwich matrix contain the identity element, which was shown in [**66**] to be sufficient for prefix-automaticity in the underlying semigroup to imply prefix-automaticity in the Rees matrix semigroup. Thus, our results improve upon some of those in [**21**] and [**66**]. For clarity, we restate our results explicitly in the semigroup case.

THEOREM 7.7. *Let $M = M^0(S; I, \Lambda; P)$ be a Rees matrix semigroup with zero (or $M = M(S; I, \Lambda; P)$ a Rees matrix semigroup without zero) over a semigroup S. Suppose there exists $\lambda \in \Lambda$ such that every non-identity element of S can be written in the form $P_{\lambda i}$ or $P_{\lambda i}s$ for some $i \in I$ and (where appropriate) $s \in S$. If M is automatic then S is automatic.*

THEOREM 7.8. *Let $M = M^0(S; I, \Lambda; P)$ be a finitely generated Rees matrix semigroup with zero (or $M = M(S; I, \Lambda; P)$ a Rees matrix semigroup without zero) over a semigroup S. Suppose there exists $\lambda \in \Lambda$ such that every non-identity element of S can be written in the form $P_{\lambda i}$ or $P_{\lambda i}s$ for some $i \in I$ and (where appropriate) $s \in S$. If S is prefix-automatic then M is prefix-automatic.*

Theorems 7.7 and 7.8 bear the same strange relationship as Theorems 9.5 and 9.7 below. See Section 9.3 below for further remarks.

A consequence of Theorem 7.8, together with some results from [**21**], is a complete characterisation of prefix-automaticity in Rees matrix semigroups over groups, and hence in completely simple and completely 0-simple semigroups.

THEOREM 7.9. *Let M be a finitely generated Rees matrix semigroup with or without zero over a group G. Then M is prefix-automatic if and only if G is automatic.*

PROOF. Suppose $M = M(G; I, \Lambda; P)$ or $M = M^0(G; I, \Lambda; P)$. If M is prefix-automatic, then by [**21**, Theorem 4.2], (or Theorem 9.8 below), G is prefix-automatic, that is, G is automatic.

Conversely, suppose G is automatic. If the sandwich matrix P is the zero matrix, then M contains no non-zero decomposable elements. For M to be finitely generated, it must then be that M is finite, and hence prefix-automatic.

Otherwise, we can choose $\lambda \in \Lambda$ and $i \in I$ such that $P_{\lambda i}$ is non-zero. Now for any element $g \in G$ we have $g = P_{\lambda i}(P_{\lambda i}^{-1} g)$. By Theorem 7.8, it follows that M is prefix-automatic, as required. $\square$

It follows also by [**15**, Theorem 1.1], [**21**, Theorem 5.3] and results of [**58**] that automaticity and prefix-automaticity coincide in completely simple and completely 0-simple semigroups, as they do in groups.

COROLLARY 7.10. *A completely simple or completely* 0*-simple semigroup is prefix-automatic exactly if it is automatic.*

CHAPTER 8

Automaticity and Partial Algebras

In this section, we introduce natural definitions of automaticity and prefix-automaticity in semigroupoids and small categories. We then show that, with these definitions, a semigroupoid is automatic or prefix-automatic exactly if the associated categorical-at-zero semigroup is automatic or prefix-automatic respectively. Echoing Chapter 3, this will allow us to extend a number of standard results about automatic semigroups to cover automatic semigroupoids.

8.1. Automaticity

A *choice of representatives* for a semigroupoid S is a triple (X, K, ρ), of a graph X with $X^0 = S^0$, a (surjective) semigroupoid morphism $\rho : X^+ \to S$ of the free semigroupoid X^+ onto S, and a language $K \subseteq X^+$ such that $K\rho = S$. The choice of representatives is called *finitely generated* if X has finitely many edges. Clearly, a semigroupoid has a finitely generated choice of representatives if and only if it is finitely generated. The choice of representatives is called a *cross-section* if the restriction of ρ to K is bijective, that is, if K contains a unique representative for every arrow in S. The choice of representatives is called *prefix-closed* if K is closed under the taking of non-empty prefixes.

A *(right-right) automatic structure* for a semigroupoid S is a finitely generated choice of representatives (X, K, ρ) with the property that for every edge or empty path $a \in X^0 \cup X^1$, the binary relation

$$K_a = \{(u, v) \in K \times K \mid u\alpha = a\omega, (ua)\rho = v\rho\}$$

is (right) synchronously regular.

We remark that, just as for semigroups (see Section 6.2), we could define analogous notions of *left-right*, *right-left*, and *left-left* automatic structures. However, we shall be concerned only with right-right automatic structures, which we shall refer to simply as automatic structures.

The following proposition gives an alternative formulation of the definition.

PROPOSITION 8.1. *A finitely generated choice of representatives* (X, K, ρ) *is an automatic structure if and only if* K_a *is synchronously regular for every edge* $a \in X^1$ *and the union*

$$K_= = \{(u, v) \in K \times K \mid u\rho = v\rho\} = \bigcup_{a \in X^0} K_a$$

is synchronously regular.

PROOF. Certainly if (X, K, ρ) is an automatic structure then K_a is synchronously regular for every $a \in X^0$, so the union $K_=$ is synchronously regular by Lemma 5.8(iv).

Conversely, if $K_=$ is synchronously regular, then for every $a \in X^0$ we have

$$K_a = \{(u, v) \in K_= \mid v\alpha = a\} = \ K_= \cap K \times X^* B$$

where

$$B = \{x \in X^1 \mid x\alpha = a\}$$

is finite. It now follows by Lemma 5.8(iii) and (iv) that each K_a is synchronously regular, and hence that (X, K, ρ) is an automatic structure. □

We shall use the equivalence of these two formulations without further comment. A consequence of this proposition is that, as in the semigroup case, it is implicit in the definition that the path language K is regular.

A *prefix-automatic structure* for a semigroupoid S is an automatic structure (X, K, ρ) with the additional property that the binary relation

$$K'_= = \{(u, v) \in K \times \mathrm{Pref}(K) \mid u\rho = v\rho\}$$

is synchronously regular. We shall see later (Proposition 8.5) that a semigroupoid has a prefix-automatic structure exactly if it has an automatic structure which is prefix-closed.

A semigroupoid is called *automatic* [*prefix-automatic*] if it has an automatic structure [respectively, a prefix-automatic structure]. An automatic [prefix-automatic] structure which is also a cross-section is called an *automatic cross-section* [respectively, a *prefix-automatic cross-section*]. We shall see (Corollary 8.7) that every automatic [prefix-automatic] semigroupoid has an automatic [respectively, prefix-automatic] cross-section.

We remarked above (Questions 6.1 and 6.3) that it is not known whether every automatic semigroup has a prefix-automatic structure, or whether every prefix-automatic semigroup has a prefix-closed automatic cross-section. These questions are also open for semigroupoids. See Section 9.3 below for further remarks.

8.2. Some Properties of Automatic Semigroupoids

The following technical result is proved exactly as in the semigroup case [**14**, Proposition 3.2].

PROPOSITION 8.2. *Suppose (X, K, ρ) is an automatic structure for a semigroupoid S, and $w \in X^+$. Then the binary relation*

$$K_w = \{(u, v) \in K \times K \mid u\alpha = w\omega, (uw)\rho = v\rho\}$$

is synchronously regular.

COROLLARY 8.3. *Let S be a semigroupoid, and $\overline{S}$ be the small category obtained by adjoining an identity at every object of S. Suppose (X, K, ρ) is an automatic structure for S, and let $s \in \overline{S}$. Then the language*

$$K_s = \{(u, v) \in K \times K \mid u\alpha = s\omega, (u\rho)s = v\rho\}$$

is synchronously regular.

PROOF. If $s \in S$, then we can choose some $w \in X^+$ with $w\rho = s$, whereupon $K_s = K_w$ is synchronously regular by Proposition 8.2. Otherwise, s is a local identity, in which case $K_s = K_a$ where a is the empty path at $s\alpha = s\omega$. By the definition of an automatic structure, it follows that K_s is synchronously regular. □

The following theorem will allow us to generalise a number of known results about automatic semigroups to the wider case of automatic semigroupoids.

THEOREM 8.4. *Let T be the categorical-at-zero semigroup associated with a semigroupoid S. Then S is automatic if and only if T is automatic. Furthermore, S has a prefix-closed automatic structure if and only if T has a prefix-closed automatic structure, and S has an automatic cross-section if and only if T has an automatic cross-section.*

PROOF. First, suppose T is automatic. Let (A, L, ρ) be an automatic structure for T, and let B be the set of letters in A which do not represent the zero element $0 \in T$. Let X be a graph with vertex set S^0, and edge set B, where for each $b \in B$, we define $b\omega = b\rho\omega$ and $b\alpha = b\rho\alpha$ to be respectively the source and target of the element of T represented by b, when considered as an arrow in S.

Let σ be the restriction of ρ to $X^+ \subseteq A^+$. Clearly, paths in X^+ represent non-zero elements of T, so that $\sigma : X^+ \to S$ is a morphism of semigroupoids.

Let

$$K = \{w \in L \mid w\rho \neq 0\}$$

be the set of words in L which do not represent the zero element $0 \in T$. It is readily verified that

$$K = L \cap X^+.$$

By Proposition 5.2, X^+ considered as a language over X^1 is regular. It follows that K is regular and then, since $K \subseteq X^+$, by Proposition 5.2 that K is a regular path language over X.

Clearly, K contains a representative for every non-zero element of T, and hence for every arrow in S. Thus, (X, K, σ) is a choice of representatives for S. Now for every $a \in X^1$, we have $K_a = L_a \cap (K \times K)$, so by Lemma 5.8(iii), Lemma 5.8(iv) and Proposition 5.7, K_a is a synchronously regular path language. Similarly, $K_= = L_= \cap (K \times K)$ is a synchronously regular path language.

We have shown that (X, K, σ) is an automatic structure for S, so that S is automatic as required. It is also clear from the method of construction that (X, K, σ) is a cross-section [a prefix-closed choice of representatives] whenever (A, L, ρ) is a cross-section [respectively, a prefix-closed choice of representatives].

Conversely, suppose (X, K, σ) is an automatic structure for the semigroupoid S. Let $A = X^1 \cup \{z\}$ be an alphabet where z is a new symbol which will represent the zero element $0 \in T$. Define a semigroup morphism $\rho : A^+ \to T$ by $x\rho = x\sigma$ for all $x \in X^1$, and $z\rho = 0$.

Let $L = K \cup \{z\} \subseteq A^+$. Then L is a regular language over A. Clearly, L maps surjectively onto T, so that (A, L, ρ) is a regular choice of representatives for T. We claim that (A, L, ρ) is an automatic structure for T.

First, since z is the only representative for the zero element $0 \in T$, we have

$$L_= = K_= \cup \{(z, z)\}$$

which, by synchronous regularity of $K_=$ and Proposition 5.7 and Lemma 5.8, is synchronously regular.

Furthermore, for any $w \in L$ we have $(wz)\rho = (w\rho)0 = 0 = z\rho$ and z is the unique representative in L for the zero element, so

$$L_z = \{(w, z) \mid w \in L\} = L \times \{z\}.$$

Since L is regular, it follows by Lemma 5.8(i) and (iii) that L_z is synchronously regular.

Now let $a \in X^1$, and suppose $(u, v) \in L_a$, so that $(ua)\rho = v\rho$. If $u, v \in K$ then it follows that $(u, v) \in K_a$. If $u = z$ then we must have $v\rho = (ua)\rho = 0$ so that $v = z$. If $v = z$ then $(ua)\rho = v\rho = 0$, so either $u = z$, or $u \in K$ and the S-product $(u\rho)(a\rho)$ is not defined, that is, $u\alpha \neq a\omega$. Thus, we have

$$L_a = K_a \cup \{(z, z)\} \cup \{(u, z) \mid u \in K, u\alpha \neq a\omega\}.$$

By a similar argument to that above, we observe that K_a and $\{(z, z)\}$ are synchronously regular. Furthermore, we have

$$\{(u, z) \mid u \in K, u\alpha \neq a\omega\} = (A^*C \cap K) \times \{z\}$$

where

$$C = \{x \in A \mid x\alpha \neq a\omega\} \subseteq A$$

is finite. It follows by Lemma 5.8(iii) that the final part of the union is synchronously regular. Thus, L_a is a finite union of synchronously regular binary relations, so by Lemma 5.8(iv), it is synchronously regular, as required.

We have shown that (A, L, ρ) is an automatic structure for T. Furthermore, one can easily verify that L is a cross-section [a prefix-closed choice of representatives] whenever (A, L, ρ) is a cross-section [respectively, a prefix-closed choice of representatives]. $\square$

The following proposition, which generalises [**65**, Proposition 5.8], will enable us to extend Theorem 8.4 to cover prefix-automatic semigroupoids.

PROPOSITION 8.5. *Let S be a semigroupoid. Then the following are equivalent:*

(i) *S is prefix-automatic.*
(ii) *S has a prefix-closed automatic structure.*
(iii) *S has a prefix-closed prefix-automatic structure.*

PROOF. Suppose first that S is prefix-automatic. Let (X, K, ρ) be a prefix-automatic structure for S. Let $L = \mathrm{Pref}(K)$. We claim that (X, L, ρ) is a (prefix-closed) automatic structure for S.

Let $\overline{S}$ be the category obtained by adjoining an identity at every object of S, and let $s \in \overline{S}$. Define

$$L_s = \{(u, v) \in L \times L \mid u\alpha = s\omega, (u\rho)s = v\rho\}$$

and similarly

$$K_s = \{(u, v) \in K \times K \mid u\alpha = s\omega, (u\rho)s = v\rho\}.$$

We show first that

$$L_s = (K'_=)^{-1} \circ K_s \circ K'_=.$$

Suppose first that $(x, y) \in L_s$. Then $x, y \in \mathrm{Pref}(K)$ and $(x\rho)s = y\rho$. Choose $z, z' \in K$ such that $z\rho = x\rho$ and $z'\rho = (x\rho)s$. Then we have $(x, z) \in (K'_=)^{-1}$, $(z, z') \in K_s$ and $(z', y) \in K'_=$, so that $(x, y) \in (K'_=)^{-1} \circ K_s \circ K'_=$. Conversely, if $(x, y) \in (K'_=)^{-1} \circ K_s \circ K'_=$ then there exist $z, z' \in K$ with $(x, z) \in (K'_=)^{-1}$, $(z, z') \in K_s$ and $(z', y) \in K'_=$. But now $x, y \in \mathrm{Pref}(K) = L$ and $(x\rho)s = (z\rho)s = z'\rho = y\rho$, so that $(x, y) \in L_s$.

Now (X, K, ρ) is a prefix-automatic structure for S, so $K'_=$ is synchronously regular, and by Lemma 5.8(i), $(K'_=)^{-1}$ is also synchronously regular. By Corollary 8.3, we also see that K_s is synchronously regular, and by Lemma 5.8(iv), it follows that L_s is synchronously regular.

In particular, it follows that $L_a = L_{1_a}$ is synchronously regular for each $a \in X^0$, and $L_a = L_{a\rho}$ is synchronously regular for each $a \in X^1$. Thus, S has a prefix-closed automatic structure. We have now shown that (i) implies (ii).

Next, suppose that S has a prefix-closed automatic structure (X, K, ρ). Then by definition, $\mathrm{Pref}(K) = K$. But then the definitions of $K_=$ and $K'_=$ coincide, so it must be that $K'_= = K_=$ is synchronously regular. Thus, (X, K, ρ) is a prefix-closed prefix-automatic structure for S. Hence, (ii) implies (iii).

Finally, that (iii) implies (i) is immediate. □

COROLLARY 8.6. *Let T be the categorical-at-zero semigroup associated with a semigroupoid S. Then S is prefix-automatic if and only if T is prefix-automatic.*

PROOF. By Proposition 8.5, S is prefix-automatic if and only if S has a prefix-closed automatic structure. By Theorem 8.4, S has a prefix-closed automatic structure exactly if T has a prefix-closed automatic structure. By Proposition 8.5, T has a prefix-closed automatic structure if and only if T is prefix-automatic. □

The above results enable us immediately to extend to semigroupoids a number of important results about semigroups and automaticity.

COROLLARY 8.7. *Let S be an automatic [prefix-automatic] semigroupoid. Then S has an automatic [respectively, prefix-automatic] cross-section.*

PROOF. Suppose first that S is automatic. Let T be the categorical-at-zero semigroup associated with S. By Theorem 8.4, T is automatic. Now by [**14**, Corollary 5.6], T has an automatic cross-section. So by Theorem 8.4, S has an automatic cross-section.

For prefix-automaticity, we argue similarly using Corollary 8.6 in place of Theorem 8.4, and the proofs of [**65**, Proposition 5.3] and [**65**, Corollary 5.4]. in place of [**14**, Corollary 5.6]. (Silva and Steinberg [**65**] prove a result analogous to [**14**, Corollary 5.6] for prefix-automaticity in monoids. However, their proof does not rely upon the existence of an identity, and suffices to prove a corresponding result for semigroups in general.) □

THEOREM 8.8. *Let S be a semigroupoid, and U be a cofinite subsemigroupoid of S. Then S is automatic [prefix-automatic] if and only if U is automatic. [respectively, prefix-automatic].*

PROOF. Let T and V be the categorical-at-zero semigroups associated with S and U respectively. Clearly, V is a cofinite subsemigroup of T. By Theorem 8.4, S is automatic if and only if T is automatic. Now by [**36**, Theorem 1.1], T is automatic if and only if V is automatic. Finally, by Theorem 8.4, V is automatic if and only if U is automatic.

For prefix-automaticity, we use the same argument with Corollary 8.6 substituted for Theorem 8.4, and Theorem 7.3 and some arguments from [**36**], in place of [**36**, Theorem 1.1]. □

We shall also need the following technical results.

PROPOSITION 8.9. *Let S be a semigroupoid with an automatic cross-section [prefix-closed automatic structure, prefix-automatic cross-section], and let T be a finite subset of S. Then S has an automatic cross-section [prefix-closed automatic structure, prefix-automatic cross-section] (Y, L, σ) such that the restriction of σ to Y is injective and has image containing T.*

PROOF. Let (X, K, ρ) be an automatic cross-section [prefix-closed automatic structure, prefix-automatic cross-section] for S. Choose a subgraph Y of X such that $Y^1\rho = X^1\rho$ and the restriction of ρ to Y^1 is injective. For each element $t \in T$ which does not already have a representative in X, adjoin a new edge y_t to Y with $y_t\omega = t\omega$ and $y_t\alpha = t\alpha$, to obtain a new finite graph Z.

For each $x \in X^1$, let $x\sigma$ be the unique element $y \in Z$ such that $y\rho = x\rho$. Extend σ to a semigroupoid morphism $\sigma' : X^+ \to (Y')^+$. Let $\rho' : Z^+ \to S$ be defined by

$$x\rho' = \begin{cases} x\rho & \text{if } x \in X \\ t & \text{if } x = y_t. \end{cases}$$

It is a routine exercise to verify that $(Z, K\sigma', \rho')$ is an automatic cross-section [prefix-closed automatic structure, prefix-automatic cross-section] for S and that the restriction of ρ' to Z is injective and has image containing T. □

LEMMA 8.10. *Let* (X, K, σ) *be a regular choice of representatives for a semigroupoid* S*, and let* L *be a cofinite subset of* K*. If* $K_= \cap (L \times L)$ *is synchronously regular and* $K_c \cap (L \times K)$ *is synchronously regular for every* $c \in X^1$*, then* (X, K, σ) *is an automatic structure for* S*.*

PROOF. First, we claim that for any $w \in K$, the language

$$w\sigma\sigma^{-1} \cap K = \{x \in K \mid x\sigma = w\sigma\}$$

is regular. If L contains no representatives for $w\sigma$, then $w\sigma\sigma^{-1} \cap K \subseteq K \setminus L$ must be finite and hence regular. Otherwise, let u be a representative in L for $w\sigma$. Now we have

$$w\sigma\sigma^{-1} \cap K = (w\sigma\sigma^{-1} \cap (K \setminus L)) \cup ((K_= \cap (L \times L)) \cap (\{u\} \times L))\pi_2$$

where $w\sigma\sigma^{-1} \cap K \setminus L \subseteq K \setminus L$ is finite, $K_= \cap (L \times L)$ is synchronously regular by assumption, and L is regular. It is follows by Lemma 5.8(ii), (iii) and (iv) that $w\sigma\sigma^{-1} \cap K$ is regular.

Now we see that

$$K_= = (K_= \cap (L \times L)) \cup \bigcup_{w \in K \setminus L} \{w\} \times (w\sigma\sigma^{-1} \cap K) \cup \bigcup_{w \in K \setminus L} (w\sigma\sigma^{-1} \cap K) \times \{w\}$$

is synchronously regular by Lemma 5.8(iii) and (iv). Similarly, for any $c \in C^1$ we see that

$$K_c = (K_c \cap (L \times K)) \cup \bigcup_{w \in K \setminus L} \{w\} \times ((wc)\sigma\sigma^{-1} \cap K)$$

is synchronously regular by Lemma 5.8(iii) and (iv). Thus, (X, K, σ) is an automatic structure for S, as required. $\square$

LEMMA 8.11. *Let* (X, L, σ) *be an automatic structure for a semigroupoid* S*, and suppose* $L \subseteq K \subseteq X^+$ *with* $K \setminus L$ *finite. Then* (X, K, σ) *is an automatic structure for* S*.*

PROOF. First, note that $K_= \cap (L \times L) = L_=$ is synchronously regular and $K_c \cap (L \times L) = L_c$ is synchronously regular for every $c \in X^1$. We can deduce as in the proof of Lemma 8.10 that $w\sigma\sigma^{-1} \cap K$ is regular for every $w \in K$, and that $K_=$ is synchronously regular.

Now let $c \in X^1$. For any $w \in K$, choose $u \in L$ such that $w\sigma = u\sigma$, and consider the set

$$\begin{aligned}\{x \in K \mid x\alpha = c\omega, (xc)\sigma = w\sigma\} =& \{x \in K \setminus L \mid x\alpha = c\omega, (xc)\sigma = w\sigma\} \\ & \cup \{x \in L \mid x\alpha = c\omega, (xc)\sigma = w\sigma\} \\ =& \{x \in K \setminus L \mid x\alpha = c\omega, (xc)\sigma = w\sigma\} \\ & \cup ((K_c \cap (L \times L)) \cap (L \times \{u\}))\pi_1.\end{aligned}$$

Certainly the set

$$\{x \in K \setminus L \mid x\alpha = c\omega, (xc)\sigma = w\sigma\}$$

is finite, and we know that $K_c \cap (L \times L)$ is synchronously regular and L is regular. It follows by Lemma 5.8(ii), (iii) and (iv) that

$$\{x \in K \mid x\alpha = c\omega, (xc)\sigma = w\sigma\}$$

is regular.

It follows that

$$\begin{aligned}K_c = (K_c \cap (L \times L)) \cup & \bigcup_{w \in K \setminus L} \{w\} \times ((wc)\sigma\sigma^{-1} \cap K) \cup \\ & \bigcup_{w \in K \setminus L} \{x \in K \mid x\alpha = c\omega, (xc)\sigma = w\sigma\} \times \{w\}\end{aligned}$$

is synchronously regular by Lemma 5.8(iii) and (iv). Thus, (X, K, σ) is an automatic structure for S, as required. □

CHAPTER 9

Automaticity and Rees Matrix Constructions

In this chapter, we consider the relationship between automaticity and prefix-automaticity in a semigroupoid, and the Rees matrix constructions introduced in Chapter 4. We extend some results of [**21**] and [**66**], to cover semigroupoids and small categories. In some cases, our new results are stronger than previously known results, even when specialised to the case of semigroups.

Much of this chapter is based on material which previously appeared in the preprint [**44**].

9.1. Automaticity

In order for a semigroup or semigroupoid to be automatic it is, of course, necessary that it be finitely generated. In Chapter 4, we found necessary and sufficient conditions for a Rees matrix semigroup over a finitely generated semigroupoid to be finitely generated. We begin by showing that the same conditions suffice to ensure that a Rees matrix semigroup over an automatic small category is automatic. We shall then extend this result to cover Rees matrix semigroups over general automatic semigroupoids.

THEOREM 9.1. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a small category S. Suppose S is automatic (as a semigroupoid) and M is finitely generated. Then M is automatic.*

PROOF. Let P' be the set of non-zero entries in the sandwich matrix P, and let $U = SP'S$. Since M is finitely generated and S is a small category, we deduce, by Theorem 4.5, that I, Λ and $S \setminus U$ are all finite. Furthermore, by Proposition 4.4, U is a subsemigroupoid of S. It follows by Theorem 8.8 that

U is automatic, by Corollary 8.7 that U has an automatic cross-section, and then by Proposition 8.9 that U has an automatic cross-section (X, K, σ) with the property that the restriction of σ to X is injective.

By the definition of U, for each $y \in X$ we can choose $s_y, t_y \in S$, $i_y \in I$ and $\lambda_y \in \Lambda$ such that $y\sigma = s_y P_{\lambda_y i_y} t_y$.

Now let

$$H = \{t_y, s_y \mid y \in X\} \cup \{1_s \mid s \in S^0\} \subseteq S.$$

We define new alphabets

$$A = \{a_{igh\lambda} \mid i \in I, g, h \in H, \lambda \in \Lambda, iF = g\omega, g\alpha = h\omega, h\alpha = \lambda G\}$$

and

$$B = \{b_{is\lambda} \mid i \in I, s \in S \setminus U, \lambda \in \Lambda, iF = s\omega, s\alpha = \lambda G\}.$$

Clearly, A and B are finite. Let z be a new symbol not in A or B, which will represent the zero element $0 \in M$. We define a morphism of semigroups

$$\rho : (A \cup B \cup \{z\})^+ \to M$$

by $(a_{igh\lambda})\rho = (i, gh, \lambda)$, $(b_{is\lambda})\rho = (i, s, \lambda)$ and $z\rho = 0$.

For every $i \in I$ and $\lambda \in \Lambda$, let $X_{i\lambda}$ be the set

$$X_{i\lambda} = \{x \in X^+ \mid iF = x\omega, x\alpha = \lambda G\}$$

of words in X^+ starting at iF and finishing at λG, and define a function

$$\phi_{i\lambda} : X_{i\lambda} \to A^+$$

by

$$(w_1 w_2 \ldots w_n)\phi_{i\lambda} = a_{i,1_{iF},s_{w_1},\lambda_{w_1}} a_{i_{w_1},t_{w_1},s_{w_2},\lambda_{w_2}} \ldots a_{i_{w_n},t_{w_n},1_{\lambda G},\lambda}$$

for all $w_1, w_2, \ldots, w_n \in X$ with $w_1 \ldots w_n \in X_{i\lambda}$. It follows from the fact that the restriction of σ to X is injective, that each function $\phi_{i\lambda}$ is injective. Indeed, if

$$(w_1 \ldots w_n)\phi_{i\lambda} = (w'_1 \ldots w'_{n'})\phi_{i\lambda}$$

then from the definition of $\phi_{i\lambda}$ we have

$$\begin{aligned} a_{i,1_{iF},s_{w_1},\lambda_{w_1}} a_{i_{w_1},t_{w_1},s_{w_2},\lambda_{w_2}} \dots a_{i_{w_n},t_{w_n},1_{\lambda G},\lambda} & \\ &= (w_1 \dots w_n)\phi_{i\lambda} \\ &= (w'_1 \dots w'_{n'})\phi_{i\lambda} \\ &= a_{i,1_{iF},s_{w'_1},\lambda_{w'_1}} a_{i_{w'_1},t_{w'_1},s_{w'_2},\lambda_{w'_2}} \dots a_{i_{w'_{n'}},t_{w'_{n'}},1_{\lambda G},\lambda} \end{aligned}$$

so clearly $n = n'$ and for $1 \leq k \leq n$ we have $s_{w_k} = s_{w'_k}$, $i_{w_k} = i_{w'_k}$, $\lambda_{w_k} = \lambda_{w'_k}$ and $t_{w_k} = t_{w'_k}$. Now each

$$w_k\sigma = s_{w_k} P_{\lambda_{w_k} i_{w_k}} t_{w_k} = s_{w'_k} P_{\lambda_{w'_k} i_{w'_k}} t_{w'_k} = w'_k\sigma.$$

But σ is by assumption injective on X, so it follows that each $w_k = w'_k$.

Note also that

$$(i, w\sigma, \lambda) = w\phi_{i\lambda}\rho \tag{9}$$

for all paths $w \in X_{i\lambda}$.

Our aim is to show that each function of the form $\phi_{i\lambda}$ is strongly regularity preserving, and that each pair of such functions is strongly mutually synchronous regularity preserving. We shall do so by showing that each such function has a regular image and that the functions have sliding window inverses which are pairwise synchronised, and then invoking Lemmas 5.9 and 5.10.

We claim first that the image $X_{i\lambda}\phi_{i\lambda}$ of each $\phi_{i\lambda}$ is regular. We say that a two-letter word

$$a_{i_1,g_1,h_1,\lambda_1} a_{i_2,g_2,h_2,\lambda_2} \in A^2$$

is *compatible* if there exists $y \in X$ such that $s_y = h_1$, $\lambda_y = \lambda_1$, $i_y = i_2$ and $t_y = g_2$, and *incompatible* otherwise. Then clearly, $X_{i\lambda}\phi_{i\lambda}$ is the set of words in $A^{\geq 2}$ which begin with a letter of the form $a_{i,1_{iF},h,\mu}$ for some $h \in H$ and $\mu \in \lambda$, end with a letter of the form $a_{j,g,1_{\lambda G},\lambda}$ for some $j \in I$ and $g \in H$, and contain no incompatible factors of length 2. Thus,

$$X_{i\lambda}\phi_{i\lambda} = (A_1 A^* A_2) \setminus (A^* A_3 A^*)$$

where

$$A_1 = \{a_{i,1_{iF},h,\mu} \in A \mid h \in H, \mu \in \Lambda, iF = h\omega, h\alpha = \mu G\},$$

$$A_2 = \{a_{j,g,1_{\lambda G},\lambda} \in A \mid j \in I, g \in H, jF = g\omega, g\alpha = \lambda G\}$$

and

$$A_3 = \{ab \in A^2 \mid ab \text{ is not compatible}\}$$

are all finite. It follows by Lemma 5.6(iii) that $X_{i\lambda}\phi_{i\lambda}$ is regular, as required.

Next, we claim that the $\phi_{i\lambda}$ have pairwise synchronised sliding window inverses. Fix $i \in I$ and $\lambda \in \Lambda$. We shall define a function

$$f : \{ab \in A^2 \mid ab \text{ is compatible}\} \to X.$$

By our observations above, the domain of f will then include all two-letter factors of words in $X_{i\lambda}\phi_{i\lambda}$.

Suppose $a_{i_1 g_1 h_1 \lambda_1} a_{i_2 g_2 h_2 \lambda_2}$ is compatible. Then by definition, there is some $y \in X$ with $s_y = h_1$, $\lambda_y = \lambda_1$, $i_y = i_2$ and $t_y = g_2$. Furthermore, we have $y\sigma = h_1 P_{\lambda_1 i_2} g_2$, and σ is injective on X, so y is the unique letter in X with this property. Thus, we can define f by the rule

$$(a_{i_1 g_1 h_1 \lambda_1} a_{i_2 g_2 h_2 \lambda})f = y.$$

to obtain a well-defined function.

We claim that $(2, f, f, f)$ is a sliding window inverse for $\phi_{i\lambda}$. To show this, suppose $w = w_1 w_2 \dots w_n \in X_{i\lambda}$ with $w_1, \dots, w_n \in X$. Then by the definition of $\phi_{i\lambda}$ we have

$$(w_1 w_2 \dots w_n)\phi_{i\lambda} = a_{i,1_{iF},s_{w_1},\lambda_{w_1}} a_{i_{w_1},t_{w_1},s_{w_2},\lambda_{w_2}} \dots a_{i_{w_n},t_{w_n},1_{\lambda G},\lambda}.$$

Let

$$y_1 = a_{i,1_{iF},s_{w_1},\lambda_{w_1}}, \quad y_2 = a_{i_{w_1},t_{w_1},s_{w_2},\lambda_{w_2}}, \quad \dots, \quad y_{n+1} = a_{i_{w_n},t_{w_n},1_{\lambda G},\lambda}$$

so that

$$(w_1 w_2 \dots w_n)\phi_{i\lambda} = y_1 y_2 \dots y_{n+1}.$$

Now from the definition of f, we see that $(y_i y_{i+1})f = w_i$ for each $1 \leq i < n$, so we have

$$(y_1y_2)f(y_2y_3)f(y_3y_4)f \dots (y_{n-1}y_n)f(y_ny_{n+1})f = w_1 \dots w_n$$

as required. It follows by Lemma 5.9 (the Sliding Window Lemma) that each $\phi_{i\lambda}$ is strongly regularity-preserving.

Furthermore, if we define $\psi : \mathbb{N} \to \mathbb{N}$ to be the constant function given by $n\psi = 1$, then

$$0\psi = 1 = |wf|$$

for any $w \in \mathrm{Pref}_2(X_{i\lambda}\phi_{i\lambda})$, and

$$m\psi = 1 = |yf|$$

for every $y \in \mathrm{Int}_2(X_{i\lambda}\phi_{i\lambda})$ which occurs starting in position m. Since the definition of ψ is independent of the choice of i and λ, it follows that ψ synchronises the sliding window inverses constructed above for any pair of functions of the form $\phi_{i\lambda}$. We have already observed that the $\phi_{i\lambda}$ are injective and have regular images, so it follows by Lemma 5.10 (the Synchronous Sliding Windows Lemma) that any pair of such functions are strongly mutually synchronous regularity preserving.

We now define a language $L \subseteq (A \cup B \cup \{z\})^+$ by

$$L = B \cup \{z\} \cup \bigcup_{i \in I, \lambda \in \Lambda} (K \cap X_{i\lambda})\phi_{i\lambda}$$

We claim that $(A \cup B \cup \{z\}, L, \rho)$ is an automatic structure (indeed, an automatic cross-section) for M.

We show first that L contains a unique representative for every element of M. Certainly $z \in L$ is the unique word representing the zero element $0 \in M$. Now consider a non-zero element $(i, s, \lambda) \in M$. If $s \in S \setminus U$ then $b_{is\lambda} \in B$ is clearly the unique word in L representing (i, s, λ), as required. Otherwise, we have $s \in U$. Now (X, K, σ) is a choice of representatives for U, so we must have $s = w\sigma$ for some path $w \in K$. Furthermore, $w\omega = s\omega = iF$ and $w\alpha = s\alpha = \lambda G$, so that $w \in X_{i\lambda}$. Now by (9) we have

$$(i, s, \lambda) = (i, w\sigma, \lambda) = w\phi_{i\lambda}\rho$$

where $w\phi_{i\lambda} \in (K \cap X_{i\lambda})\phi_{i\lambda} \subseteq L$. Furthermore, if $i' \in I$ and $\lambda' \in \Lambda$ and $v \in K \cap X_{i'\lambda'}$ are such that $v\phi_{i'\lambda'} = (i, s, \lambda)$, then by (9) we must have $(i', v\sigma, j') = (i, w\sigma, j)$ from which it follows that $i = i'$, $j = j'$ and, since K contains a unique representative for every element in U, $v = w$. Hence $w\phi_{i\lambda}$ is the unique representative for (i, s, λ) in L. Thus, $(A \cup B \cup \{z\}, L, \rho)$ is a cross-section for M.

Next, we show that L is a regular language. Clearly, B and $\{z\}$ are finite and hence regular. Furthermore, K is regular and each $\phi_{i\lambda}$ is strongly regularity-preserving, so each $(K \cap X_{i\lambda})\phi_{i\lambda}$ is regular. It follows that L is a finite union of regular languages, and hence is regular.

Since $(A \cup B \cup \{z\}, L, \rho)$ is a cross-section for M, it follows by Lemma 5.8(v) that the binary relation

$$L_= = \{(u, v) \in L \times L \mid u\rho = v\rho\} = \{(u, u) \mid u \in L\}$$

is synchronously regular.

It remains to show that L_a is synchronously regular for all $a \in A \cup B \cup \{z\}$. First, since z is the unique representative in L for the zero element $0 \in M$, we have

$$L_z = \{(w, z) \mid w \in L\} = L \times \{z\}$$

which is synchronously regular by Lemma 5.8(iii).

Now suppose $a \in A \cup B$, with $a\rho = (i_a, s_a, \lambda_a)$. By Lemma 8.10, it will suffice to show that

$$L_a \cap ((A^+ \cap L) \times L) = L_a \cap (A^+ \times L)$$

is synchronously regular. We write $L_a \cap (A^+ \times L)$ as a union

$$\begin{aligned} L_a \cap (A^+ \times L) =& (L_a \cap (A^+ \times B)) \\ & \cup (L_a \cap (A^+ \times \{z\})) \\ & \cup (L_a \cap (A^+ \times A^+)), \end{aligned}$$

and show that each component in this union is synchronously regular.

Suppose $(u, v) \in L_a \cap (A^+ \times L)$, so that $(u\rho)(a\rho) = v\rho$. Clearly $v \notin B$, since letters in B represent indecomposable elements of M, so $L_a \cap (A^+ \times B)$ is empty and hence synchronously regular.

If $v = z$, then we have $(u\rho)(a\rho) = v\rho = z\rho = 0$. Since $u \in A^+ \cap L$ cannot represent the zero element, this can happen only if $P_{u\rho\pi_3, a\rho\pi_1} = 0$. Thus,

$$\begin{aligned} L_a \cap (A^+ \times \{z\}) &= \{(u, z) \mid u \in A^+, P_{u\rho\pi_3, a\rho\pi_1} = 0\} \\ &= \left(\bigcup_{b \in A, P_{(b\rho\pi_3), a\rho\pi_1} = 0} (A^* b \cap L) \right) \times \{z\} \end{aligned}$$

is a product of finite unions of regular languages, and by Lemma 5.8(iii) is synchronously regular.

It remains only to show that $L_a \cap (A^+ \times A^+)$ is synchronously regular. Certainly we have

$$L_a \cap (A^+ \times A^+) = \bigcup_{i \in I, \lambda \in \Lambda} L_{a,i,\lambda}$$

where each

$$L_{a,i,\lambda} = \{(u, v) \in L_a \mid u, v \in A^+, u\rho\pi_1 = v\rho\pi_1 = i, u\rho\pi_3 = \lambda\}.$$

If $P_{\lambda,(a\rho\pi_1)} = 0$ then $L_{a,i,\lambda}$ is empty and hence synchronously regular. Otherwise, let w be a path in K such that $w\sigma = P_{\lambda,(a\rho\pi_1)}(a\rho\pi_2)$. Then by Proposition 8.2, the language K_w is regular. Now using (9) we have

$$\begin{aligned} L_{a,i,\lambda} &= \{(u,v) \in (K \cap X_{i\lambda})\phi_{i\lambda} \times (K \cap X_{i,a\rho\pi_3})\phi_{i,a\rho\pi_3} \mid (u\rho)(a\rho) = v\rho\} \\ &= \{(u,v) \in (K \cap X_{i\lambda})\phi_{i\lambda} \times (K \cap X_{i,a\rho\pi_3})\phi_{i,a\rho\pi_3} \mid (u\rho\pi_2)P_{\lambda,a\rho\pi_1}(a\rho\pi_2) = v\rho\pi_2\} \\ &= \{(x\phi_{i\lambda}, y\phi_{i,a\rho\pi_3}) \mid (x,y) \in K_w \cap (X_{i\lambda} \times X_{i,a\rho\pi_3})\}. \end{aligned}$$

But $\phi_{i\lambda}$ and $\phi_{i,a\rho\pi_3}$ are strongly mutually synchronous regularity preserving, so it follows that $L_{a,i,\lambda}$ is synchronously regular. Hence, L_a is a finite union of synchronously regular binary relations, and by Lemma 5.8(iv) is synchronously regular as required. □

This result extends easily from small categories to semigroupoids.

THEOREM 9.2. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a semigroupoid S. Suppose S is automatic and M is finitely generated. Then M is automatic.*

PROOF. Let P' be the set of non-zero entries in the sandwich matrix P and let $U = SP'S$. Since M is finitely generated, Theorem 4.5 tells us that I, Λ and $S \setminus U$ are all finite. Let $\overline{S}$ be the category obtained from S by adjoining a new identity arrow 1_e at every object $e \in S^0$ (even if there is already an identity arrow at e). Since I and Λ are finite, the images IF and ΛG of F and G are finite. But since S is isolation-free, it follows from Proposition 4.1 that S^0 is the union of IF and $G\Lambda$, so S has only finitely many objects. It follows that S is a cofinite subsemigroupoid of $\overline{S}$ so, by Theorem 8.8, $\overline{S}$ is automatic.

Define

$$\overline{I} = I \cup \{i_s \mid s \in S^0 \setminus IF\}$$

and extend F to a function $\overline{F} : \overline{I} \to S^0$ by defining $i_s\overline{F} = s$ for all $s \in S^0 \setminus IF$. Similarly, define

$$\overline{\Lambda} = \Lambda \cup \{\lambda_s \mid s \in S^0 \setminus \Lambda G\}$$

and extend G to a function $\overline{G} : \overline{\Lambda} \to S^0$ by defining $\lambda_s\overline{G} = s$ for all $s \in S^0 \setminus \Lambda G$. Extend P to a $\overline{\Lambda} \times \overline{I}$ matrix $\overline{P}$, by defining all new entries to be zero.

Now let $\overline{M} = M^0(\overline{S}, \overline{F}(\overline{I}), \overline{G}(\overline{\Lambda}); \overline{P})$. We know that $\overline{I}$ and $\overline{\Lambda}$ are finite and that $\overline{S}$ is automatic. Furthermore, the set of non-zero entries in $\overline{P}$ is exactly the set P' of non-zero entries in P. Now if we let $\overline{U} = \overline{S}\ P'\ \overline{S}$, then we have $U \subseteq \overline{U}$, from which it follows that $\overline{S} \setminus \overline{U}$ is finite. It follows by Theorem 9.1 that $\overline{M}$ is automatic.

Now clearly every non-zero element of M is also contained in $\overline{M}$. Moreover, the multiplication in M is clearly the same as that in $\overline{M}$, so we conclude that M is a subsemigroup of $\overline{M}$. Furthermore, the only elements of $\overline{M}$ not in M are those of the form $(i, 1_e, \lambda)$ for $i \in \overline{I}$, $\lambda \in \overline{\Lambda}$ and $e \in S^0$ with

$$i\overline{F} = 1_e\omega = e = 1_e\alpha = \lambda\overline{G}.$$

Since S has finitely many objects and $\overline{I}$ and $\overline{\Lambda}$ are finite it follows that M is a cofinite subsemigroup of $\overline{M}$. By Theorem 8.8 (or [**35**, Theorem 1.1]), it follows that M is automatic. □

Combining Theorem 9.2 with Theorem 4.3, we obtain a more explicit sufficient condition, without reference to M being finitely generated.

Corollary 9.3. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a semigroupoid S. If*

(i) *the indexing sets I and Λ are finite;*
(ii) *the semigroupoid S is automatic; and*
(iii) *$S \setminus U$ is finite*

then M is automatic.

We remark briefly on a possible alternative proof strategy for the preceding results. One can extend [**66**, Theorem 6.3], which says that every Rees quotient of an automatic monoid by a regular ideal is automatic, to cover general semigroups. We could employ this result, in place of Proposition 4.2, in an argument similar to the proof of Theorem 4.3 above. Our new results would then arise as a corollary of the corresponding results for semigroups [**21**, Theorems 3.1 and 3.2]. However, the work saved would be limited, since much of the work in the above direct proofs will be re-used in proving Theorem 9.7 below. That result is stronger than any previously known for semigroups, and so cannot be expected to arise as a corollary of existing semigroup results in the same manner.

Next, we give a sufficient condition for the underlying semigroupoid of an automatic Rees matrix semigroup to be automatic.

Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a semigroupoid S. Let T be a subset of S. We say that T is *strongly right-ideal-generated by a row cross-section of* P if there exists a subset $\Lambda' \subseteq \Lambda$ such that the restriction of G to Λ' is bijective, and every arrow in T can be written in the form $P_{\lambda i}s$ for some $\lambda \in \Lambda'$, $i \in I$ and $s \in S^1$. We say that T is *(weakly) right-ideal-generated by a row cross-section of* P if there exists Λ' as above, such that every arrow in T can be written either in the form $P_{\lambda i}s$ or in the form $P_{\lambda i}$ (or both) for some $\lambda \in \Lambda'$, $i \in I$ and (where appropriate) $s \in S^1$. For a discussion of this condition in the semigroup (single-object) case, see Section 7.3.

The next result says that the set of non-identity elements of S being strongly right-ideal-generated by a row cross-section of P is a sufficient condition for automaticity in M to imply automaticity in S. We shall subsequently strengthen the result by weakening the hypothesis, showing that it suffices for the set of non-identity elements of S to be *weakly* right-ideal-generated by a row cross-section of P.

THEOREM 9.4. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a semigroupoid S. Suppose M is automatic, and the set of non-identity elements of S is strongly right-ideal-generated by a row cross-section of P. Then S is automatic.*

PROOF. It follows from Corollary 8.7 (or [**14**, Corollary 5.6]) that M has an automatic cross-section and then from Proposition 8.9 that M has an automatic cross-section (A, L, ρ) with the property that the restriction of ρ to A is injective. Let $\Lambda' \subseteq \Lambda$ be such that the restriction of G to Λ' is bijective, and every non-identity arrow in S can be written in the form $P_{\lambda i}s$ for some $\lambda \in \Lambda'$, $i \in I$ and $s \in S^1$.

Choose some subset $I' \subseteq I$ such that F restricts to a bijection on I'. Let $F' : S^1\omega \to I'$ and $G' : S^1\alpha \to \Lambda'$ be the inverses of the restrictions of F and G to I' and Λ' respectively. We define sets

$$C = \{c_s \mid s \in S, (i, s, \lambda) \in A\rho \text{ for some } i \in I, \lambda \in \Lambda\}$$

and

$$D = \{d_{\lambda i} \mid \lambda \in \Lambda, i \in I, P_{\lambda i} \neq 0\}.$$

Clearly, C and D are finite. Let X be a graph with vertex set S^0, and edge set $C \cup D$ where $c_s\omega = s\omega$, $c_s\alpha = s\alpha$, $d_{\lambda i}\omega = P_{\lambda i}\omega$ and $d_{\lambda i}\alpha = P_{\lambda i}\alpha$. Then there is a natural morphism $\sigma : X^+ \to S$, given by $c_s\sigma = s$ and $d_{\lambda i}\sigma = P_{\lambda i}$.

Let V be the language

$$V = (I' \times S \times \Lambda')\rho^{-1}$$

of words in A^+ which represent non-zero elements of the form $(i, s, \lambda) \in M$ with $i \in I'$ and $\lambda \in \Lambda'$. Define a function $\phi : V \to X^+$ by

$$(a_1 a_2 \dots a_n)\phi = c_{s_1} d_{\lambda_1 i_2} c_{s_2} d_{\lambda_2 i_3} \dots d_{\lambda_{n-1} i_n} c_{s_n}$$

where each $a_k\rho = (i_k, s_k, \lambda_k)$. That ϕ is a well-defined function into X^+ follows from the fact that words in V represent non-zero elements of M. Furthermore, if

$$(a_1 a_2 \dots a_n)\phi = c_{s_1} d_{\lambda_1 i_2} c_{s_2} d_{\lambda_2 i_3} \dots d_{\lambda_{m-1} i_m} c_{s_m} = (b_1 b_2 \dots b_{n'})\phi$$

then clearly $n = m = n'$ and for $1 \leq k \leq n$ we have

$$a_k \rho = (i_k, s_k, \lambda_k) = b_k \rho$$

where $i_1 = s_1 \omega F'$ and $\lambda_m = s_m \alpha G'$. But ρ is by assumption injective when restricted to the alphabet A, so it follows that ϕ is injective.

Notice also that for any $w \in V$ we have

$$w\rho = (w\phi\sigma\omega F', w\phi\sigma, w\phi\sigma\alpha G'). \tag{10}$$

Our aim is to show that the function ϕ has a regular image and a self-synchronised sliding window inverse. We claim first that the image $V\phi$ of the function ϕ is regular. We say that an ordered triple $(d_{\lambda i}, c_s, d_{\mu j}) \in D \times C \times D$ is a *valid internal triple* if there exists a letter $a \in A$ with $a\rho = (i, s, \mu)$. We say that $(c_s, d_{\lambda i}) \in C \times D$ is a *valid start pair* if there exists $a \in A$ with $a\rho = (i', s, \lambda)$ for some $i' \in I'$. We say that $(d_{\lambda i}, c_s) \in D \times C$ is a *valid end pair* if there exists $a \in A$ with $a\rho = (i, s, \lambda')$ for some $\lambda' \in \Lambda'$. Now it is easily verified that

$$V\phi \cap X^{\geq 5} = Q_1 (CD)^* C Q_2 \setminus (C \cup D)^* Q_3' (C \cup D)^*$$

where

$$Q_1 = \{cd \in CD \mid (c, d) \text{ is a valid start pair}\},$$

$$Q_2 = \{dc \in DC \mid (d, c) \text{ is a valid end pair}\}, \text{ and}$$

$$Q_3' = \{cde \in DCD \mid (c, d, e) \text{ is } \textit{not} \text{ a valid internal triple}\}$$

are all finite. It follows that $V\phi \cap X^{\geq 5}$ is regular, and hence that $V\phi$ is regular, as required.

Next, we claim that ϕ has a self-synchronised sliding window inverse. Consider a three-letter prefix $v_1v_2v_3$ of a word

$$(a_1a_2\dots a_n)\phi \in V\phi.$$

We shall define

$$(v_1v_2v_3)f = a_1.$$

To show that f is well-defined, suppose $(b_1b_2\dots b_m)\phi$ also has the prefix $v_1v_2v_3$. Then by the definition of ϕ we have

$$(a_1a_2\dots a_n)\phi = c_{s_1}d_{\lambda_1 i_2}c_{s_2}d_{\lambda_2 i_3}\dots d_{\lambda_{n-1}i_n}c_{s_n}$$

where each $a_k\rho = (i_k, s_k, \lambda_k)$, and similarly

$$(b_1b_2\dots b_m)\phi = c_{t_1}d_{\mu_1 j_2}c_{t_2}d_{\mu_2 j_3}\dots\dots\dots d_{\mu_{m-1}j_m}c_{t_m}$$

where each $b_k\rho = (j_k, t_k, \mu_k)$. But now we have $c_{s_1} = v_1 = c_{t_1}$ and $d_{\lambda_1 i_2} = v_2 = d_{\mu_1 j_2}$. But by the definitions of C and D, it follows that $s_1 = t_1$ and $\lambda_1 = \mu_1$. Furthermore, we must have $i_1F = s_1\omega = t_1\omega = j_1F$, but $i_1, j_1 \in I'$ and F is injective on I', so we must have $i_1 = j_1$. Thus, we obtain

$$a_1\rho = (i_1, s_1, \lambda_1) = (j_1, t_1, \mu_1) = b_1\rho.$$

But ρ is, by assumption, injective on the alphabet A, so we must have $a_1 = b_1$, as required to show that f is well-defined.

Similarly, given a word

$$(a_1a_2\dots a_n)\phi = c_{s_1}d_{\lambda_1 i_2}c_{s_2}d_{\lambda_2 i_3}\dots d_{\lambda_{n-1}i_n}c_{s_n}$$

we define

$$(d_{\lambda_{p-1}i_p}c_{s_p}d_{\lambda_p i_{p+1}})g = a_p$$

and

$$(c_{s_p}d_{\lambda_p i_{p+1}}c_{s_{p+1}})g = \epsilon$$

where ϵ denotes the empty word in A^*. A similar argument to that for f shows that g is well-defined.

Finally, given a three-letter suffix $v_1v_2v_3$ of a word

$$(a_1a_2\dots a_n)\phi \in V\phi.$$

We shall define

$$(v_1v_2v_3)h = a_n.$$

Once again, a similar argument to that for f shows that h is well-defined.

Now suppose $w = a_1 \dots a_n \in V$ and $w\phi = y_1 \dots y_m$. Then by the definition of ϕ, we have

$$y_1 \dots y_m = (a_1a_2 \dots a_n)\phi = c_{s_1} d_{\lambda_1 i_2} c_{s_2} d_{\lambda_2 i_3} \dots d_{\lambda_{n-1} i_n} c_{s_n}$$

where each $a_k\rho = (i_k, s_k, \lambda_k)$. But now by the definitions of f, g and h, it follows that

$$\begin{aligned}(y_1y_2y_3)f(y_2y_3y_4)g(y_3y_4y_5)g \dots (y_{m-3}y_{m-2}y_{m-1})g(y_{m-2}y_{m-1}y_m)h \\ = a_1a_2\epsilon a_3\epsilon \dots \epsilon a_{n-1}a_n = w\end{aligned}$$

We have shown that $(3, f, g, h)$ is a sliding window inverse for ϕ, and by Lemma 5.9 (the Sliding Window Lemma), it follows that ϕ is strongly regularity preserving.

Furthermore, if we define a function $\psi : \mathbb{N} \to \mathbb{N}$ by

$$n\psi = \begin{cases} 1 & \text{if } n = 0 \text{ or } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

then since factors of paths in $V\phi$ begin with a letter from C exactly if they begin in an odd position, ψ synchronises $(3, f, g, h)$ with itself. We have already observed that ϕ is injective and has a regular image. It follows by Lemma 5.10 (the Synchronous Sliding Windows Lemma) that ϕ is strongly synchronous regularity preserving.

We now define $K = (L \cap V)\phi \subseteq X^+$, and claim that (A, K, σ) is an automatic structure (indeed, an automatic cross-section) for S.

First, we show that σ maps K bijectively onto S. To this end, let $s \in S$. Then there is an element $m = (s\omega F', s, s\alpha G') \in M$, so there exists a word $w \in L$ representing m. Indeed, since $s\omega F' \in I'$ and $s\alpha G' \in \Lambda'$, we have $w \in V$, and so $w\phi \in K$. But by (10), $w\phi\sigma = w\rho\pi_2 = s$, so $w\phi$ is a representative in K for s.

Furthermore, if $v\phi \in (L \cap V)\phi = K$ also represents s, then we must have $v\rho = (i', s, \lambda')$ where $i' \in I'$ and $\lambda' \in \Lambda'$. But for $(i', s, \lambda') \in M$, we must have $i'F = s\omega = iF$ and $\lambda' G = s\alpha = \lambda G$. Since F and G are bijective when restricted to I' and Λ', we must have $i = i'$ and $\lambda = \lambda'$, and so $v\rho = w\rho$. But $v, w \in L$ and (A, L, ρ) is a cross-section for M, so we deduce that $v = w$, and hence $v\phi = w\phi$. Thus, (X, K, σ) is a cross-section for S.

We know that L is a regular language, and that ϕ is strongly regularity-preserving, so it is immediate that $K = (L \cap V)\phi$ is a regular language. Since (X, K, σ) is a cross-section for S, it follows that the binary relation

$$K_= = \{(u, v) \in K \times K \mid u\sigma = v\sigma\} = \{(u, u) \mid u \in K\}$$

is synchronously regular by Lemma 5.8(v).

Now let b be an edge in X. We must show that

$$K_b = \{(u, v) \in K \times K \mid u\alpha = b\omega, (ub)\sigma = v\sigma\}$$

is synchronously regular.

If $b\sigma$ is an identity arrow, then it follows easily from the fact that $K_=$ is synchronously regular that K_b is synchronously regular, and we are done.

Otherwise, $b\sigma$ is not an identity arrow in S. Now by assumption, we can write $b\sigma = P_{\lambda i}c$ for some $\lambda \in \Lambda'$, $i \in I$ and $c \in S$. Furthermore, we must have $b\omega = b\sigma\omega = (P_{\lambda i}c)\omega = \lambda G$, so that $\lambda = b\omega G'$. Let $\mu = b\alpha F'$. Now M has an element (i, c, μ). By Corollary 8.3, the language

$$L_{(i,c,\mu)} = \{(u, v) \in L \times L \mid (u\rho)(i, c, \mu) = v\rho\}$$

is synchronously regular.

We claim that

$$K_b = \{(u\phi, v\phi) \mid (u, v) \in L_{(i,c,\mu)} \cap (V \times V), u\rho\pi_3 = \lambda\}.$$

It will then follow that

$$K_b = \{(u\phi, v\phi) \mid (u, v) \in L_{(i,c,\mu)} \cap (A^*D \times A^+) \cap (V \times V)\}$$

where

$$D = \{a \in A \mid a\rho\pi_3 = \lambda\}$$

is finite, so that A^*D is regular by Lemma 5.8. By Lemma 5.8(iii) and (iv) it will follow that $L_{(i,c,\mu)} \cap (A^*D \times A^*)$ is synchronously regular. Finally, since ϕ is strongly synchronous regularity preserving, we shall deduce that K_b is synchronously regular, as required.

To prove the claim, first suppose that $(a_1, a_2) \in K_b$. Then certainly $(a_1, a_2) \in K = (L \cap V)\phi$, so $a_1 = u_1\phi$ and $a_2 = u_2\phi$ for some $u_1, u_2 \in L \cap V$. Now by (10) we have

$$u_k\rho = (u_k\phi\sigma\omega F', u_k\phi\sigma, u_k\phi\sigma\alpha G')$$

for $k = 1$ and $k = 2$. Now we have

$$u_1\rho\pi_3 = u_1\phi\sigma\alpha G' = a_1\sigma\alpha G' = a_1\alpha G' = b\omega G' = \lambda.$$

Similarly, we obtain

$$u_1\rho\pi_1 = u_1\phi\sigma\omega F' = a_1\sigma\omega F' = a_2\sigma\omega F' = u_2\phi\sigma\omega F' = u_2\rho\pi_1$$

and

$$u_2\rho\pi_3 = u_2\phi\sigma\alpha G' = a_2\sigma\alpha G' = b\sigma\alpha G' = \mu = (i, c, \mu)\pi_3.$$

Finally, we have

$$u_2\rho\pi_2 = u_2\phi\sigma = a_2\sigma = (a_1\sigma)(b\sigma) = (a\sigma)P_{\lambda i}c = ((u_1\rho)(i,c,\mu))\pi_2.$$

We have shown that $(u_1\rho)(i,c,\mu)$ and $(u_2\rho)$ are equal in all three components and certainly $u_1, u_2 \in L \cap V$, from which it follows that $(u_1, u_2) \in L_{(i,c,\mu)}$ as required.

Conversely, suppose $(u_1, u_2) \in L_{(i,c,\mu)} \cap (V \times V)$, and that $u_1\rho\pi_3 = \lambda$. Then $u_2\rho = (u_1\rho)(i,c,\mu)$, and using (10) and equating second components, we get

$$\begin{aligned} u_2\phi\sigma &= u_2\rho\pi_2 \\ &= (u_1\rho\pi_2)P_{u_1\rho\pi_3,i}c \\ &= (u_1\phi\sigma)P_{\lambda i}c \\ &= (u_1\phi\sigma)(b\sigma). \end{aligned}$$

By now by the definition of K_b, it follows that $(u_1\phi, u_2\phi) \in K_b$.

It follows, as discussed above, that K_b is synchronously regular, as required to complete the proof. □

This result extends to cover the case where S is only weakly right-ideal-generated by a row cross-section of P.

THEOREM 9.5. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a semigroupoid S. Suppose M is automatic, and the set of non-identity elements of S is weakly right-ideal-generated by a row cross-section of P. Then S is automatic.*

PROOF. Since M is automatic it is certainly finitely generated, so Theorem 4.5 tells us that S is finitely generated and that the indexing sets I and Λ are finite. Let $\overline{S}$ be the category obtained from S by adjoining a new identity arrow at every object $e \in S^0$ which does not already have one. Define $\overline{I}$,

$\overline{\Lambda}$, $\overline{P}$, $\overline{M}$ and $\overline{U}$ just as in the proof of Theorem 9.2, again noting that $\overline{I}$ and $\overline{\Lambda}$ are finite.

Still reasoning as in the proof of Theorem 9.2, we deduce that M is a cofinite subsemigroupoid of $\overline{M}$. It follows by Theorem 8.8 (or [**36**, Theorem 1.1]) that $\overline{M}$ is automatic. We also deduce that $\overline{S} \setminus \overline{U}$ is finite.

Next, we wish to show that the set of non-identity elements of $\overline{S}$ is strongly right-ideal-generated by a row cross-section of $\overline{P}$. Since the non-identity elements of S are weakly right-ideal-generated by a row cross-section of P, we can choose some set $\Lambda' \subseteq \Lambda$ such that the restriction of G to Λ' is bijective, and every non-identity element of S can be written in the form $P_{\lambda i}c$ or $P_{\lambda i}$ for some $\lambda \in \Lambda'$, $i \in I$ and (where appropriate) $c \in S$. Let

$$\overline{\Lambda}' = \Lambda' \cup (\overline{\Lambda} \setminus \Lambda).$$

Now suppose s is a non-identity element of $\overline{S}$. Notice that, because we have adjoined an identity only where S did not already have one, every identity in S remains an identity in $\overline{S}$. Thus, we can assume that s is a non-identity element of S. Thus, there exist $\lambda \in \Lambda'$ and $i \in I$ such that either $s = P_{\lambda i}c$ for some $c \in S \subseteq \overline{S}$, or $s = P_{\lambda i} = P_{\lambda i}e$ where $e \in \overline{S}$ is the identity at $P_{\lambda i}\alpha = iF$. Thus, the set of non-identity elements of $\overline{S}$ is strongly right-ideal-generated by a row cross-section of P.

It now follows by Theorem 9.4 that $\overline{S}$ is automatic. But as in the proof of Theorem 9.2, we deduce that S is a cofinite subsemigroupoid of $\overline{S}$. Now by Theorem 8.8, it follows that S is automatic, as required. $\square$

9.2. Prefix-Automaticity

The following result, which provides a sufficient condition for a Rees Matrix semigroup over a semigroupoid to be prefix-automatic, generalises [**66**, Theorem 7.2] in the case of prefix-automaticity. As with Theorems 9.4 and 9.5, we shall prove this result first with the hypothesis that the semigroupoid S is a small category and that the non-identity elements of S are strongly right-ideal-generated by a row cross-section of the sandwich matrix P. We shall then extend the result to the more general case in which S is a semigroupoid with non-identity elements weakly right-ideal-generated by a row cross-section of P.

THEOREM 9.6. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a finitely generated Rees matrix semigroup with zero over a small category S. If S is prefix-automatic, and the non-identity elements of S are strongly right-ideal-generated by a row cross-section of P, then M is prefix-automatic.*

PROOF. Let $\Lambda' \subseteq \Lambda$ be such that the restriction of G to Λ' is bijective, and every non-identity arrow in S can be written in the form $P_{\lambda i}s$ for some $\lambda \in \Lambda'$, $i \in I$ and $s \in S$.

We define U as in the proof of Theorem 9.1, once again deducing from the fact that M is finitely generated, that I, Λ and $S \setminus U$ are all finite and that U is a subsemigroupoid of S. It follows by Theorem 8.8 that U is prefix-automatic, and by Proposition 8.5 that U has a prefix-closed automatic structure. Note that since S is finitely generated, it has finitely many objects and hence finitely many local identities. It follows by Proposition 8.9 that U has a prefix-closed automatic structure (X, K, σ) with the property that the restriction of ρ to X is injective. (Note that, since we require our automatic structure to be prefix-closed, we cannot insist that it should also be a cross-section.)

We define A, B, z, L, H, ρ and each $X_{i\lambda}$ and $\phi_{i\lambda}$ exactly as in the proof of Theorem 9.1, and claim now that $(A \cup B \cup \{z\}, L, \rho)$ is a prefix-automatic structure for S.

By exactly the same arguments as in the proof of Theorem 9.1, we deduce that $(A \cup B \cup \{z\}, L, \rho)$ is a regular choice of representatives for M, and that L_a is synchronously regular for all letters $a \in A \cup B \cup \{z\}$. However, since (X, K, σ) is not here assumed to be a cross-section for S, we cannot deduce that $(A \cup B \cup \{z\}, L, \rho)$ is a cross-section for M, and we must work a little harder to show that $L_=$ is regular.

By Lemma 8.10, it will suffice to show that

$$L_= \cap (A^+ \times A^+)$$

is synchronously regular. Now certainly

$$L_= \cap (A^+ \times A^+) = \bigcup_{i \in I, \lambda \in \Lambda} L_{i,\lambda}$$

where

$$L_{i,\lambda} = \{(u,v) \in L_= \cap (A^+ \times A^+) \mid u\rho\pi_1 = v\rho\pi_1 = i, u\rho\pi_3 = v\rho\pi_3 = \lambda\}.$$

But

$$\begin{aligned} L_{i,\lambda} &= \{(u,v) \in (K \cap X_{i\lambda})\phi_{i\lambda} \times (K \cap X_{i\lambda})\phi_{i\lambda} \mid u\rho = v\rho\} \\ &= \{(x\phi_{i\lambda}, y\phi_{i\lambda}) \mid (x,y) \in K_= \cap (X_{i\lambda} \times X_{i\lambda})\}. \end{aligned}$$

From our argument in the proof of Theorem 9.1, we know that $\phi_{i\lambda}$ is strongly synchronous regularity preserving, and we know that $K_=$ is synchronously regular, so it follows that each $L_{i,\lambda}$ is synchronously regular. Hence, $L_= \cap (A^+ \times A^+)$ is a finite union of synchronously regular binary relations, and by Lemma 5.8(iv) is synchronously regular as required.

Thus, we conclude that $(A \cup B \cup \{z\}, L, \rho)$ is an automatic structure for M. It remains only to show that the language

$$L'_= = \{(u,v) \mid u \in L, v \in \mathrm{Pref}(L), u\rho = v\rho\}$$

is synchronously regular. First, observe that we can write

$$\begin{aligned} L'_{=} = {} & (L'_{=} \cap (B \times \mathrm{Pref}(L))) \\ & \cup (L'_{=} \cap (L \times B)) \\ & \cup (L'_{=} \cap (\{z\} \times \mathrm{Pref}(L)) \\ & \cup (L'_{=} \cap (L \times \{z\})) \\ & \cup (L'_{=} \cap (A^+ \times A^+)). \end{aligned}$$

Clearly, z and letters in B are unique representatives in $\mathrm{Pref}(L)$ for the respective elements they represent. Thus, we have

$$L'_{=} \cap (B \times \mathrm{Pref}(L))) = L'_{=} \cap (L \times B) = \{(b, b) \mid b \in B\}$$

and

$$L'_{=} \cap (\{z\} \times \mathrm{Pref}(L)) = L'_{=} \cap (L \times \{z\}) = \{(z, z)\}$$

so that the first four components of the union are synchronously regular.

It remains to show that $L'_{=} \cap (A^+ \times A^+)$ is synchronously regular. Define

$$\overline{L} = L \cup \{a_{i1_s1_s\lambda} \mid i \in I, s \in S^0, \lambda \in \Lambda, iF = s = \lambda G\}.$$

Clearly, since I and Λ are finite and S has finitely many objects, $\overline{L} \setminus L$ is finite. It follows by Lemma 8.11 that $\{A \cup B \cup \{z\}, \overline{L}, \rho)$ is an automatic structure for M.

Given any letter $a = a_{ist\lambda} \in A$, we define $\overline{a} = a_{is1_{s\alpha}\lambda'}$, where $\lambda' = s\alpha G'$ is the unique element in Λ' satisfying $\lambda' G = s\alpha$.

Now consider a non-empty (not necessarily proper) prefix $a_1 \ldots a_k$ of a word $a_1 \ldots a_n$ in $L \cap A^+$. We claim that $a_1 \ldots a_{k-1}\overline{a_k} \in \overline{L}$. If $k = 1$ then $\overline{a_k}$ is of the form $a_{i1_s1_s\lambda}$, and so is by definition in $\overline{L}$. Otherwise, we observe that from the definitions of L and $\phi_{i\lambda}$ we must have

$$a_1 \ldots a_n = (y_1 \ldots y_{n-1})\phi_{i\lambda}$$

for some $i \in I$, $\lambda \in \Lambda$ and some word $y_1 \dots y_{n-1} \in K \cap X_{i\lambda}$. But K is prefix-closed, so also $y_1 \dots y_{k-1} \in K$. Furthermore, if we let $\mu = y_{k-1}\alpha$ then we have $y_1 \dots y_{k-1} \in K \cap X_{i\mu}$ and

$$(y_1 \dots y_{k-1})\phi_{i\mu} = a_1 \dots a_{k-1}\overline{a_k}.$$

Thus, $a_1 \dots a_{k-1}\overline{a_k} \in L \subseteq \overline{L}$ as claimed.

Recall from page 109 the definitions of the subset $H \subseteq S$ and the alphabet A. Let $s \in H$ and $\lambda \in \Lambda$ be such that $s\alpha = \lambda F$. We consider separately the cases in which s is and is not an identity element in S.

First, suppose that s is not an identity element. Then by assumption, we can choose $\lambda' \in \Lambda'$, $i_s \in I$ and $t_s \in S$ such that $s = P_{\lambda' i_s} t_s$. Then certainly $i_s F = t_s \omega$ and $t_s \alpha = s\alpha = \lambda F$, so there exists an element $(i_s, t_s, \lambda) \in M$.

Since we have shown that $(A \cup B \cup \{z\}, \overline{L}, \rho)$ is an automatic structure for M, we deduce by Proposition 8.3 that the language

$$\overline{L}_{(i_s,t_s,\lambda)} = \{(u, v) \in \overline{L} \mid (u\rho)(i_s, t_s, \lambda) = v\rho\}$$

is synchronously regular.

We define a new language

$$\begin{aligned} N_{s,\lambda} = \{(u, a_1 \dots a_k) \mid & u \in L, a_1 \dots a_k \in \mathrm{Pref}(L), a_k = a_{i,s',s,\lambda}, \\ & (a_1 \dots a_{k-1}\overline{a_k}, u) \in \overline{L}_{(i_s,t_s,\lambda)}\} \end{aligned}$$

By Corollary 5.12 and Lemma 5.8(i), $N_{s,\lambda}$ is synchronously regular.

Now for any $u \in L$ and $a_1 \dots a_k \in \mathrm{Pref}(L)$ with a_k of the form $a_{i,s',s,\lambda}$ we have

$$\overline{a_k}\rho\pi_3 = s'\alpha G' = s\omega G' = P_{\lambda' i_s}\omega G' = \lambda'$$

so that

$$\begin{aligned}(u, a_1 \dots a_k) \in N_{s,\lambda} &\iff (a_1 \dots a_{k-1}\overline{a_k}, u) \in \overline{L}_{(i_s,t_s,\lambda)}\\ &\iff (a_1 \dots a_{k-1}\overline{a_k})\rho(i_s, t_s, \lambda) = u\rho\\ &\iff (a_1 \dots a_{k-1})\rho(i, s'P_{\lambda' i_s}t_s, \lambda) = u\rho\\ &\iff (a_1 \dots a_{k-1})\rho(i, s's, \lambda) = u\rho\\ &\iff (a_1 \dots a_k)\rho = u\rho\\ &\iff (u, a_1 \dots a_k) \in L'_{=}.\end{aligned}$$

Next, we consider the case in which s is a local identity in S. In this case, we define

$$\begin{aligned}N_{s,\lambda} = \{(b_1 \dots b_n, a_1 \dots a_k) \mid a_1 \dots a_k \in \operatorname{Pref}(L), b_1 \dots b_n \in L,\\ a_k = a_{i,s',s,\lambda}, b_n = a_{j,t,1_{t\omega},\lambda}, (a_1 \dots a_{k-1}\overline{a_k}, b_1 \dots b_{n-1}\overline{b_n}) \in \overline{L}_{=}\}.\end{aligned}$$

Since $(A \cup B \cup \{z\}, \overline{L}, \rho)$ is an automatic structure for M, we deduce that $\overline{L}_{=}$ is regular, and then by Corollary 5.12 and Lemma 5.8(i) that $N_{s,\lambda}$ is synchronously regular.

Now for any $a = a_1 \dots a_k \in \operatorname{Pref}(L)$ and $b = b_1 \dots b_n \in L$ with a_k of the form $a_{i,s',s,\lambda}$ and b_n of the form $b_{j,t,1_{t\omega},\lambda}$ we have

$$\begin{aligned}(b, a) \in N_{s,\lambda} &\iff (a_1 \dots a_{k-1}\overline{a_k}, b_1 \dots b_{n-1}\overline{b_n}) \in \overline{L}_{=}\\ &\iff (a_1 \dots a_{k-1}\overline{a_k})\rho = (b_1 \dots b_{n-1}\overline{b_n})\rho\\ &\iff (a_1 \dots a_{k-1})\rho(i, s', \lambda GG') = (b_1 \dots b_{n-1})\rho(j, t, \lambda GG')\\ &\iff (a_1 \dots a_{k-1})\rho(i, s's, \lambda) = (b_1 \dots b_{n-1})\rho(j, t1_{t\omega}, \lambda)\\ &\iff (a_1 \dots a_k)\rho = (b_1 \dots b_n)\rho\\ &\iff (b_1 \dots b_n, a_1 \dots a_k) \in L'_{=}.\end{aligned}$$

It follows from the two cases considered that

$$L'_{=} = \bigcup_{s \in H, \lambda \in \Lambda} N_{s,\lambda}$$

so that $L'_{=}$ is a union of finitely many synchronously regular languages, and hence by Lemma 5.8(iv) is synchronously regular. □

We now extend this result as described above.

THEOREM 9.7. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a finitely generated Rees matrix semigroup with zero over a semigroupoid S. If S is prefix-automatic, and the set of non-identity elements in S is weakly right-ideal-generated by a row cross-section of P, then M is prefix-automatic.*

PROOF. We combine the methods used to prove Theorems 9.2 and 9.5.

As in the proof of Theorem 9.2, we deduce that the indexing sets I and Λ are finite. We define $\overline{S}$, $\overline{I}$, $\overline{\Lambda}$, $\overline{P}$, $\overline{M}$ and $\overline{U}$ as in the proof of Theorem 9.5, noting that $\overline{S}$ has an adjoined identity only where there was not already an identity in S.

As in Theorem 9.5, we deduce that S is a cofinite subsemigroupoid of $\overline{S}$, that M is a cofinite subsemigroup of $\overline{M}$, and that the non-identity elements of $\overline{S}$ are strongly right-ideal-generated by a row cross-section of $\overline{P}$.

It follows by Theorem 8.8 that $\overline{S}$ is prefix-automatic, by Theorem 9.6 that $\overline{M}$ is prefix-automatic and then by Theorem 8.8 (or Theorem 7.3) that M is automatic, as required. □

Implications of this result for Rees matrix semigroups over semigroups and groups in particular, are discussed in Section 7.3 above.

We now turn our attention to showing that automaticity in a Rees matrix semigroup is a sufficient condition for automaticity in the underlying semigroupoid. The following result generalises [**21**, Theorem 4.2].

THEOREM 9.8. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a semigroupoid S. If M is prefix-automatic then S is prefix-automatic.*

PROOF. Suppose M is prefix-automatic. Then by Proposition 8.5, M has a prefix-closed automatic structure, and by Proposition 8.9, M has a prefix-closed automatic structure (A, L, ρ) with the property that the restriction of ρ to A is injective. (Once again, we note that, because we require an automatic structure which is prefix-closed, we cannot ask also that it be a cross-section.)

Choose some subsets $I' \subseteq I$ and $\Lambda' \subseteq \Lambda$ such that F and G restrict to bijections on I' and Λ' respectively. Define F', G', C, D, X, σ, V, ϕ and K exactly as in the proof of Theorem 9.4. Reasoning as before, we deduce that ϕ is strongly regularity preserving and strongly synchronous regularity preserving, and that for any $w \in V$ we have

$$w\rho = (w\phi\sigma\omega F', w\phi\sigma, w\phi\sigma\alpha G'). \tag{11}$$

We deduce also that (X, K, σ) is a regular choice of representatives for S, although we can no longer conclude that it is a cross-section. We claim that (X, K, σ) is a prefix-automatic structure for S.

Because of the limited role played by the alphabet in the definition of an automatic structure, we can assume without loss of generality that for every letter $a \in A$, there is a letter in A representing the element

$$(a\rho\pi_1, a\rho\pi_2, a\rho\pi_3 GG') \in M.$$

For each $a \in A$, let $\overline{a} \in A$ be such a letter.

Let $c \in X^1 \cap X^0$ be an edge or a path of length 0 in X. Consider the binary relation

$$K'_c = \{(x, y) \in \mathrm{Pref}(K) \times K \mid x\alpha = c\omega, (xc)\sigma = y\sigma\}.$$

Our aim is to show that K'_c is synchronously regular. We claim that

$$K'_c = \left(\bigcup_{\lambda \in \Lambda, i \in I, b \in C \cup CD} K_{\lambda,i,b} \right) \cup \left(\bigcup_{x \in \mathrm{Pref}(K) \cap X^{\leq 4}} K'_c \cap (\{x\} \times K) \right) \tag{12}$$

where each

$$K_{\lambda,i,b} = \{(((w_1 \dots w_{n-1}\overline{w_n})\phi)d_{\lambda i}b, v\phi) \mid w_1, w_2, \dots, w_n \in A, v \in L \cap V$$
$$w = w_1 \dots w_n \in L, w\rho\pi_3 = \lambda, (w_1 \dots w_n, v) \in L_{(i,(bc)\sigma,c\alpha G')}\}.$$

To prove the claim, suppose first that $(x, y) \in K'_c$. Clearly, if $|x| \leq 4$ then, since $(x, y) \in K'_c \cap (\{x\} \times K)$, we see that (x, y) is contained in the right-hand-side of (12) and we are done.

Now suppose $|x| \geq 5$. Certainly $y \in K = (L \cap V)\phi$, so we can write $y = v\phi$ for some $v \in L \cap V$. Also $x \in \mathrm{Pref}(K) = \mathrm{Pref}((L \cap V)\phi)$, so certainly we can choose $w \in L \cap V$ and $z \in X^*$ with $xz = w\phi$. From the definition of ϕ, we have $xz \in C(DC)^*$. Since $|x| \geq 5$, it follows that we can write $x = ad_{\lambda i}b$ for some $a \in CDC(DC)^*$, $\lambda \in \Lambda$, $i \in I$ and either $b \in C$ or $b \in CD$.

Suppose $w = w_1 \dots w_k$. Then it is easily verified from the definition of ϕ (see page 118) that

$$a = (w_1 \dots w_{n-1}\overline{w_n})\phi$$

where $n = (|a| - 1)/2$. Notice also that we must have $w\rho\pi_3 = \lambda$. Now we have

$$\begin{aligned}
(w\rho)(i, (bc)\sigma, c\alpha G') &= (w\rho\pi_1, u\rho\pi_2, \lambda)(i, (bc)\sigma, c\alpha G') \\
&= (u\rho\pi_1, a\sigma, \lambda)(i, (bc)\sigma, c\alpha G') \\
&= (u\rho\pi_1, (a\sigma)P_{\lambda i}(bc)\sigma, c\alpha G') \\
&= (u\rho\pi_1, (ad_{\lambda i}bc)\sigma, c\alpha G') \\
&= (v\rho\pi_1, (xc)\sigma, v\rho\pi_3) \\
&= (v\rho\pi_1, y\sigma, v\rho\pi_3) \\
&= v\rho
\end{aligned}$$

so that $(w, v) \in L_{(i,(bc)\sigma,c\alpha G')}$ and hence $((w_1 \dots w_{n-1}\overline{w_n})\phi d_{\lambda i} b, v\phi) \in K_{\lambda,i,b}$.

Conversely, suppose $w = w_1 \dots w_n \in L$ with

$$((w_1 \dots w_{n-1}\overline{w_n}\phi)d_{\lambda i} b, v\phi) \in K_{\lambda,i,b}$$

for some $\lambda \in \Lambda, i \in I$ and $s \in S$. Let

$$a = (w_1 \dots w_{n-1}\overline{w_n})\phi.$$

Then we have

$$\begin{aligned}
(v\rho\pi_1, (((w_1 \dots w_{n-1}\overline{w_n})\phi)d_{\lambda i} bc)\sigma, v\rho\pi_3) &= (u\rho\pi_1, (ad_{\lambda i} bc)\sigma, c\alpha G') \\
&= (u\rho\pi_1, (a\sigma)P_{\lambda i}(bc)\sigma, c\alpha G') \\
&= (u\rho\pi_1, a\sigma, \lambda)(i, (bc)\sigma, c\alpha G') \\
&= (w\rho\pi_1, w\rho\pi_2, \lambda)(i, (bc)\sigma, c\alpha G') \\
&= (w\rho)(i, (bc\sigma), c\alpha G') \\
&= v\rho
\end{aligned}$$

so in particular we have

$$(((w_1 \dots w_{n-1}\overline{w_n})\phi)d_{\lambda i} b\sigma)(c\sigma) = v\phi\sigma$$

so that

$$(((w_1 \dots w_{n-1}\overline{w_n})\phi)d_{\lambda i} b\sigma, v\phi\sigma) \in K'_c$$

as required.

Also, it is clear that if $x \in \mathrm{Pref}(K) \cap X^{\leq 4}$ and $y \in K$ are such that $(x, y) \in K'_c \cap (\{x\} \times K)$ then we must have $(x, y) \in K'_c$. Thus, we have justified our claim that (12) holds.

Now by Corollary 8.3, each binary relation $L_{(i,(bc)\sigma,c\alpha G')}$ is synchronously regular. Furthermore, the functions given by

$$A \to A, a \mapsto \overline{a}$$

and

$$X \to X^{|b|+2}, x \mapsto xd_{\lambda i}b$$

both satisfy the conditions of Corollary 5.12. Since ϕ is also synchronous regularity preserving, it follows that each $K_{\lambda,i,b}$ is synchronously regular.

Now let $x \in \mathrm{Pref}(K) \cap X^{\leq 4}$. Choose a word u in L representing the element $(x\omega F', (x\sigma)(c\sigma), c\alpha G')$. Now for any word $v\phi \in (L \cap V)\phi = K$, we have $(xc)\sigma = v\phi\sigma$ if and only if $u\rho = v\rho$, which in turn is true exactly if $(u, v) \in L_=$. Thus, we have

$$K'_c \cap (\{x\} \times K) = \{x\} \times ((\{u\} \times L) \cap L_=)\pi_2\phi.$$

We know that L is regular and that $L_=$ is synchronously regular, so it follows by Lemma 5.8(ii), (iii) and (iv) and the fact that ϕ is strongly synchronous regularity preserving that $K'_c \cap (\{x\} \times K)$ is synchronously regular.

We have shown that each K'_c is a finite union of synchronously regular binary relations, and it follows by Lemma 5.8(iv) that each K'_c is synchronously regular. Now for every $c \in X^1 \cup X^0$ we see that

$$K_c = K'_c \cap (K \times K)$$

is synchronously regular by Lemma 5.8(iv), so that (X, L, σ) is an automatic structure for S. Moreover,

$$K'_= = \bigcup_{c \in X^0} (K'_c)^{-1}$$

is also synchronously regular by Lemma 5.8(iv), so that (X, K, σ) is a prefix-automatic structure for S, as required. $\square$

9.3. Remarks

The curious relationship between the results of Section 9.2 and those of Section 9.1 seems to demand comment. In the case of automaticity, showing that automaticity in a Rees matrix semigroup is a *sufficient* condition for automaticity in the underlying semigroupoid requires a right-ideal-generation condition, while the converse implication does not. In the case of prefix-automaticity, the situation is entirely reversed – the right-ideal-generation condition is required only to show that prefix-automaticity in a Rees matrix semigroup is a *necessary* condition for prefix-automaticity in the underlying semigroupoid.

We have already remarked (Question 6.1) that it is an open question whether every automatic semigroup (or every automatic semigroupoid) is prefix-automatic. Theorem 8.4 and Corollary 8.6 together imply that this question is no harder in the ostensibly more general semigroupoid context, that is, that every automatic semigroupoid is prefix-automatic, exactly if every automatic semigroup is prefix-automatic. However, the following questions do naturally arise.

QUESTION 9.9. *Does there exist an automatic Rees matrix semigroup over a non-automatic semigroup?*

QUESTION 9.10. *Does there exist an automatic Rees matrix semigroup over a non-automatic semigroupoid?*

Clearly, if every automatic semigroup (and hence every automatic semigroupoid) is prefix-automatic, then Theorem 9.8 gives a negative answer to both of these questions. However, if there *are* automatic semigroups (and hence semigroupoids) which are not prefix-automatic, then the answers to Questions 9.9 and 9.10 could be both positive, both negative, or negative and positive respectively.

9.4. Rees Matrix Semigroups Without Zero

Theorem 8.8 ensures that our results about Rees matrix semigroups with zero adapt easily to the case of Rees matrix semigroups without zero.

THEOREM 9.11. *Let $M = M(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup (without zero) over a semigroupoid S. Suppose S is automatic and M is finitely generated. Then M is automatic.*

PROOF. Consider the Rees matrix semigroup with zero

$$M' = M^0(S; F(I), G(\Lambda); P).$$

Then $M = M' \setminus \{0\}$ is a cofinite subsemigroup of M'. So by Theorem 3.5 (or [**62**, Theorem 1.1]), M' is finitely generated. Now by Theorem 9.2, M' is automatic, and by Theorem 8.8 (or [**36**, Theorem 1.1]) it follows that M is automatic. □

THEOREM 9.12. *Let $M = M(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup (without zero) over a semigroupoid S. Suppose the set of non-identity elements of S is weakly right-ideal-generated by a row cross-section of P. If M is automatic then S is automatic.*

PROOF. Consider the Rees matrix semigroup with zero

$$M' = M^0(S; F(I), G(\Lambda); P).$$

Then $M = M' \setminus \{0\}$ is a cofinite subsemigroupoid of M'. Now if M is automatic then by Theorem 8.8, M' is automatic, and by Theorem 9.5, S is automatic. □

THEOREM 9.13. *Let $M = M(S; F(I), G(\Lambda); P)$ be a finitely generated Rees matrix semigroup (without zero) over a semigroupoid S. If S is prefix-automatic, and the set of non-identity elements of S is weakly right-ideal-generated by a row cross-section of P, then M is prefix-automatic.*

PROOF. Consider the Rees matrix semigroup with zero

$$M' = M^0(S; F(I), G(\Lambda); P).$$

Then $M = M' \setminus \{0\}$ is a cofinite subsemigroup of M'. So by Theorem 3.5 (or [**62**, Theorem 1.1]), M' is finitely generated. Now by Theorem 9.7, M' is prefix-automatic, and by Theorem 8.8 (or Theorem 7.3) it follows that M is prefix-automatic. $\square$

THEOREM 9.14. *Let $M = M(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup (without zero) over a semigroupoid S. If M is prefix-automatic then S is prefix-automatic.*

PROOF. Consider the Rees matrix semigroup with zero

$$M' = M^0(S; F(I), G(\Lambda); P).$$

Then $M = M' \setminus \{0\}$ is a cofinite subsemigroupoid of M'. Now if M is prefix-automatic then by Theorem 8.8, M' is prefix-automatic. By Theorem 9.8, it follows that S is prefix-automatic. $\square$

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