PRESENTATIONS FOR SEMIGROUPS AND SEMIGROUPOIDS

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Abstract. We consider the relationship between the combinatorial properties of semigroupoids in general and semigroups in particular. We show that a semigroupoid is finitely generated [finitely presentable] exactly if the corresponding categorical-at-zero semigroup is finitely generated [respectively, finitely presentable]. This allows us to extend some of the main results of [17], to show that finite generation and presentability are preserved under finite extension of semigroupoids and the taking of cofinite subsemigroupoids. We apply this result to extend the results of [6], giving characterisations of finite generation and finite presentability in Rees matrix semigroups over arbitrary semigroupoids.

1. Introduction

Recently, a number of authors have considered generators and relations for semigroups, and in particular whether the properties of finite generation and presentability are preserved under various constructions. A survey of recent work in this area can be found in [1], while more details appear in [2, 3, 4, 6, 7, 8, 9, 13, 16].

It has long been known that the basic notions of generators and relations can be generalised to categories [12, 15], with graphs and paths substituting for alphabets and words respectively. The same approach applies naturally to semigroupoids. In this paper, we show that some of the recent theory of semigroup presentations generalises to the corresponding notions in semigroupoids (and small categories).

These results have implications for the combinatorial theory not only of semigroupoids in general, but also of semigroups in particular. Certain classes of semigroups are naturally described by means of constructions over semigroupoids and small categories. Perhaps best known are certain semigroups which are *categorical at zero* [10, Section 7.7]. More recently, Lawson [14] has introduced a method of finding abundant semigroups, which employs a form of Rees matrix construction over semigroupoids. We show that, under certain assumptions, a semigroup constructed in either of these ways is finitely generated or finitely presented exactly if the underlying semigroupoid is, respectively, finitely generated or finitely presented.

2000 *Mathematics Subject Classification.* 20M05.
Key words and phrases. semigroups, semigroupoids, generators, relations, presentations, Rees matrix semigroups.

In addition to this introduction, this paper comprises six sections. In Section 2 we introduce concepts and notation which will be used throughout the paper. In Section 3 we show that a semigroupoid is finitely generated or presentable exactly if the associated categorical-at-zero semigroup is, respectively, finitely generated or presentable. This allows us to extend a result of [17], showing that the properties of finite generation and presentability are preserved under finite extension and the taking of cofinite subsemigroupoids. In Sections 4, 5, 6 and 7 we consider the preservation of the same properties under Rees matrix constructions of semigroups over semigroupoids, extending some results of [6].

2. Preliminaries

In this section, we briefly introduce the concepts and notation used throughout this paper.

2.1. **Graphs, Semigroupoids and Categories.**

For our purposes, a *(directed) graph* X consists of a collection V of *vertices*, a collection E of *edges* and two functions $\alpha, \omega : E \to V$ which determine respectively the *source* and *target* of each edge. Note that we allow multiple edges joining the same pair of vertices, and edges which start and end at the same vertex. We say that X is *small* if V and E are sets, and *finite* if V and E are finite sets.

A *path* in X is a finite sequence $e_1 e_2 \dots e_n$ of (not necessarily distinct) edges in X such that $\alpha(e_i) = \omega(e_{i+1})$ for $1 \leq i < n$. The source, target and *length* of a path are defined in the obvious way. We also define a distinct *empty path* of length 0 at each vertex, with source and target that vertex.

A *semigroupoid* (S, S_0) is a small graph with vertex set S_0 and edge set S, together with a partial multiplication on the edges of S such that, for any two edges e and f

(i) the product ef is defined if and only if $\alpha(e) = \omega(f)$;
(ii) if ef is defined, then $\omega(ef) = \omega(e)$ and $\alpha(ef) = \alpha(f)$; and
(iii) if ef and fg are defined then $e(fg) = (ef)g$.

The vertices and edges of a semigroupoid are called respectively *objects* and *arrows*. Where the meaning is clear from the context, we shall speak of "the semigroupoid S" rather than "the semigroupoid (S, S_0)".

If $s_0 \in S_0$ is such that there exists an $s \in S$ with $\alpha(s) = \omega(s) = s_0$, we say that the *local semigroup* of S at the object s_0 is the semigroup with elements $\{s \in S \mid \alpha(s) = \omega(s) = s_0\}$ and multiplication defined by restricting that in S. A semigroupoid is *non-degenerate* if it has a local semigroup at every object, that is, if for every object $s_0 \in S_0$, there exists an arrow $s \in S$ such that $\alpha(s) = \omega(s) = s_0$. A semigroupoid is *strongly connected* if for every pair $s_0, t_0 \in S_0$ of objects, there is an arrow (or equivalently, a path) with source s_0 and target t_0.

There is a natural multiplication on the subsets of S, given by

$$AB = \{ab \mid a \in A, b \in B, \alpha(a) = \omega(b)\}$$

for all $A, B \subseteq S$. A *subsemigroupoid* of a semigroupoid (S, S_0) is a semigroupoid (U, S_0) where $U \subseteq S$ and the multiplication on U is a restriction of that on S. Often, we shall just say that "U is a subsemigroupoid of S".

Let (S, S_0) and (T, S_0) be semigroupoids with the same set of objects. A function $\sigma : S \to T$ is called a *(semigroupoid) homomorphism* if $\alpha \circ \sigma = \alpha$, $\omega \circ \sigma = \omega$ and $\sigma(s)\sigma(t) = \sigma(st)$ for all $s, t \in S$ with $\alpha(s) = \omega(t)$.

An arrow e is called an *identity* at the object $\alpha(e)$ if $\alpha(e) = \omega(e)$, and $ef = f$ for all arrows f such that ef is defined, and $fe = f$ for all arrows f such that fe is defined. There is, necessarily, at most one identity at a given object. A semigroupoid with an identity at every object is called a *small category*. We write 1_e to denote the unique identity at the object $e \in S_0$. For a semigroupoid to be a category, it is necessary (but not sufficient) that each of its local semigroups be a monoid. An arrow e is said to be an *inverse* of an arrow f if ef and fe are both defined and equal the identities at $\alpha(f)$ and $\alpha(e)$ respectively. A small category in which every arrow has an inverse is called a *groupoid*.

2.2. **Categorical-at-zero Semigroups.** Let S be a semigroupoid and 0 be a new symbol not in S. Let $T = S \cup \{0\}$, and define a multiplication on T by

$$st = \begin{cases} \text{the } S\text{-product } st & \text{if } s, t \in S \text{ and } \alpha(s) = \omega(t) \\ 0 & \text{otherwise} \end{cases}$$

for all $s, t \in T$. Clearly, T is a semigroup. Also, T is *categorical at zero* [10, Section 7.7], that is, for any elements $a, b, c \in T$, if ab and bc are non-zero then abc is also non-zero. We shall call T the *categorical-at-zero semigroup associated with S*.

2.3. **Free Semigroupoids and Presentations.** A semigroupoid (S, S_0) is *finitely generated* if there exists a finite subset $Y \subseteq S$ such that every arrow in S can be written as a product of arrows in Y.

The *free semigroupoid* X^+ on a small graph X is the semigroupoid whose objects are the vertices of X, and whose arrows are the **non-empty** paths in X, with α, ω and the partial multiplication defined in the obvious way. The *free category* X^* is the category obtained by adjoining an identity (which can be thought of as the empty path) at each object of X^+.

Let S be a semigroupoid. A *congruence* on S is an equivalence relation ρ on the arrows of S such that

(i) if $e\rho f$ then $\alpha(e) = \alpha(f)$ and $\omega(e) = \omega(f)$; and
(ii) if $e\rho f$, $g\rho h$ and eg and fh are defined then $(eg)\rho(fh)$.

We denote by S/ρ the semigroupoid with object set S_0 and arrows the equivalence classes of ρ, with multiplication defined in the obvious way.

Suppose R is a binary relation on S which satisfies property (i). It is straightforward to show that there is a unique minimum congruence $R^{\#}$ on S which contains R.

Let S be a semigroupoid, X be a small graph, and R be a binary relation on the free semigroupoid X^+, which satisfies property (i). We say that $\langle X \mid R \rangle$ is a presentation for S, if S is isomorphic to $X^+/R^{\#}$. Every semigroupoid has a presentation, and it is easy to show that a semigroupoid is finitely generated exactly if it has a presentation $\langle X \mid R\rangle$ such that X is a finite graph. A presentation $\langle X \mid R\rangle$ is called *finite* if X and R are finite, and a semigroupoid with a finite presentation is called *finitely presentable.*

A pair $(u, v) \subseteq X^+ \times X^+$ of paths with the same start and end point is called a *relation.* Given a homomorpism from X^+ onto S, the relation (u, v) is said to be *satisfied* in S if u and v *represent* (map onto) the same element in S. Two paths $x, y \in X^+$ are said to be *related by an application* of the relation (u, v) if $x = sut$ and $y = svt$ for some $s, t \in X^*$. If $R \subseteq X^+ \times X^+$ then x and y are *related as a consequence of* R if there exist paths $x = p_1, p_2, p_3, \dots, p_n = y$ such that each p_i is related to p_{i+1} by an application of some relation in R. It is clear that x and y are related as a consequence of R exactly if $(x, y) \in R^{\#}$. Indeed, given a graph X and a homomorphism from the free semigroupoid X^+ onto a semigroupoid S, one can show that $\langle X \mid R\rangle$ is a presentation for S (with respect to the given homomorphism) exactly if every relation in R is satisfied in S and every relation in $X^+ \times X^+$ which is satisfied in S is a consequence of relations in R.

To relate these definitions to the usual concepts in combinatorial semigroup theory, we observe that non-degenerate, single-object semigroupoids are in a natural bijective correspondence with their respective (unique) local semigroups. Extending this correspondence to presentations, we see that single-vertex small graphs correspond to *alphabets* and paths in such graphs to *words.* It is a routine exercise to verify that, in the single object case, our other definitions simplify to correspond with those usual in semigroup theory.

2.4. **Rees Matrix Semigroups over Semigroupoids.** Rees matrix semigroup constructions over semigroupoids were introduced by Lawson [14], who used them to construct a class of locally adequate abundant semigroups. They represent an alternative formulation of certain special cases of the *blocked Rees matrix semigroup* constructions introduced by Fountain [11] and subsequently employed by Armstrong [5]. We extend Lawson's definition in a natural way to define Rees matrix semigroups *with zero* over semigroupoids. This slightly more general construct reformulates Fountain's approach in the general case.

Let S be a non-degenerate semigroupoid, and I and Λ be indexing sets. Let $F : I \to S_0$ and $G : \Lambda \to S_0$ be surjective functions. Let P be a $\Lambda \times I$ matrix with entries drawn from $S \cup \{0\}$, with the property that $\alpha(P_{\lambda i}) = F(i)$ and $\omega(P_{\lambda i}) = G(\lambda)$ for all $i \in I, \lambda \in \Lambda$ such that $P_{\lambda i} \neq 0$. The *Rees matrix semigroup with zero* $M = M^0(S; F(I), G(\Lambda); P)$ is the semigroup with set of elements

$$M = \{0\} \cup \{(i, x, \lambda) \in I \times S \times \Lambda \mid F(i) = \omega(x), \alpha(x) = G(\lambda)\}$$

and multiplication given by

$$(i, x, \lambda)(j, y, \mu) = \begin{cases} (i, xP_{\lambda j}y, \mu) & \text{if } P_{\lambda j} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $(i, x, \lambda), (j, y, \mu) \in M \setminus \{0\}$, and

$$0m = m0 = 0$$

for all $m \in M$.

If P contains no zero entries, then $M \setminus \{0\}$ is a subsemigroup of M, which we call the *Rees matrix semigroup (without zero)* $M(S; F(I), G(\Lambda); P)$. Clearly, such a construction can be defined only over a strongly connected semigroupoid.

We observe that every Rees matrix semigroup over a semigroupoid is a subsemigroup of a Rees matrix semigroup over the associated categorical-at-zero semigroup.

3. Semigroupoids and Categorical-at-zero Semigroups

The following easy theorem will allow us to generalise a number of known results about semigroup presentations to the wider case of semigroupoid presentations.

Theorem 1. *Let T be the categorical-at-zero semigroup associated with a semigroupoid S. Then S is a finitely generated [finitely presentable] semigroupoid if and only if T is a finitely generated [finitely presentable] semigroup.*

Proof. First, suppose S is finitely generated by a subset $Y \subseteq S$. Then it is easy to see that the finite set $Y \cup \{0\}$ generates T. Conversely, if T is finitely generated by a subset $Z \subseteq T$, then the finite set $Z \cap S$ generates S.

Now suppose S has a finite presentation $\langle X \mid R \rangle$. Let A be the set of edges of X, and let $\phi : X^+ \to A^+$ be defined in the obvious way. Let z be a symbol not in X which will represent the zero element in T. Let

$$\begin{aligned} R' &= \{(\phi(v), \phi(w)) \mid (v, w) \in R\} && (1)\\ &\cup \{(za, z), (az, z) \mid a \in A \cup \{z\}\} && (2)\\ &\cup \{(ab, z) \mid a, b \in A, \alpha(a) \neq \omega(b)\} && (3)\end{aligned}$$

We claim that

$$\langle A \cup \{z\} \mid R' \rangle$$

is a finite presentation for T with the obvious isomorphism. It is straightforward to verify that R' is finite and that all the relations in R' hold in T, and it follows from our argument above that T is generated by the image of $A \cup \{z\}$. It remains to show that all relations which hold in T are consequences of the relations in R'.

Suppose $a = a_1 a_2 \ldots a_m$ and $b = b_1 b_2 \ldots b_n$ are words in $(A \cup \{z\})^+$ such that the relation (a, b) holds in T. Let $t \in T$ be the element represented by a (and by b). First, consider the case where $t = 0$. If a contains the symbol z, then a is related by a chain of applications of relations from (2) to z. Otherwise, $a \in A^+$ but clearly a does not represent a path in X. It follows that there must exist $i < m$ such that $\alpha(a_i) \neq \omega(a_{i+1})$. Now a is related by a relation from (3) to $a' = a_1 \ldots a_{i-1} z a_{i+2} \ldots a_m$, which in turn is related by a chain of applications of relations from (2) to z. A similar argument shows that b is related by a chain of applications of relations in R' to z, and thence to a.

Otherwise, if $t \neq 0$ then a and b correspond to paths in X, which represent the element t as an arrow of S. It follows easily that a and b are represented by a chain of applications of relations from (1), as required.

Conversely, suppose $\langle A \mid R\rangle$ is a finite presentation for T. Let B be the set of letters in A which do not represent zero. For each $b \in B$, define $\alpha(b)$ and $\omega(b)$ to be respectively the source and target of the element of T represented by b, when considered as an arrow in S. Let X be the graph with vertex set S_0 and edge set B, with the source and target functions α and ω as just defined. Let $\phi : X^+ \to A^+$ be defined in the obvious way.

We observe that the image $\phi(X^+)$ of ϕ is exactly the set of words in A^+ which do not represent 0. Clearly, since $\{0\}$ is an ideal of T, $\phi(X^+)$ is closed under the taking of factors. Also, if $\phi(X^+)$ contains a representative for an element $x \in T$, then it contains all representatives in A^+ for x. It follows that the only relations in R which can be applied to words in $\phi(X^+)$ are those in $R \cap (\phi(X^+) \times \phi(X^+))$. With this in mind, we define a set of relations

$$R' = \{(v, w) \mid (\phi(v), \phi(w)) \in R\}$$

and claim that $\langle X \mid R\rangle$ is a finite presentation for A with the obvious isomorphism. It is easy to verify that S is generated by the image of X, and that relations in R' are satisfied in S. Now suppose a relation (v, w) holds in S. Then certainly also the relation $(\phi(v), \phi(w))$ holds in T, so $\phi(v)$ and $\phi(w)$ are related by a chain of applications of relations in R. But by our observations above, these relations must all be contained in $\phi(X^+) \times \phi(X^+)$, and so have counterparts in R'. It follows easily that v and w are related by a chain of applications of relations in R'. □

This result allows us immediately to extend to semigroupoids one of the most important results about finite generation and presentation of semigroups.

Theorem 2. *Let S be a semigroupoid, and $U \subseteq S$ be a subsemigroupoid of S such that $S \setminus U$ is finite. Then S is finitely generated [finitely presentable] if and only if U is finitely generated [respectively, finitely presentable].*

Proof. Let T and V be the categorical-at-zero semigroups associated with S and U respectively. Clearly, V is a cofinite subsemigroup of T. Now by Theorem 1, S is finitely generated [finitely presentable] if and only if T is finitely generated [finitely presentable]. By Theorem 1.1 [Theorem 1.3] of [17], T is finitely generated [finitely presentable] if and only if V is finitely generated [finitely presentable]. Finally, by Theorem 1, V is finitely generated [finitely presentable] if and only if U is finitely generated [finitely presentable]. □

The following result, the semigroup analogue of which is well known [10, Section 9.2], says that a finitely presentable semigroupoid has a finite presentation with respect to every finite choice of generators.

Theorem 3. *Let S be a finitely presentable semigroupoid, and Y be a finite graph such that there exists a natural surjective homomorphism from Y^+ onto S. Then there exists a finite set of relations $R' \subseteq Y^+ \times Y^+$ such that $\langle Y \mid R'\rangle$ is a (finite) presentation for S.*

Proof. Suppose S has a finite presentation $\langle X \mid R\rangle$. For each edge $y \in Y$, let $\lambda(y) \in X^+$ be a path in X representing the same arrow as y. Similarly, for each edge $x \in X$, let $\rho(x) \in Y^+$ be a path in Y representing the same arrow as x. Define a set of relations

$$\begin{aligned} R' &= \{(\rho(x_1)\rho(x_2)\dots\rho(x_m), \rho(x_1')\dots\rho(x_n')) \mid (x_1\dots x_m, x_1'\dots x_n') \in R\} \\ &\cup \{(y, \rho(x_1)\dots\rho(x_n)) \mid y \in Y, \lambda(y) = x_1\dots x_n\}. \end{aligned}$$

It is an easy exercise to show that $\langle Y \mid R' \rangle$ is a finite presentation for S. □

4. Finite Generation of Rees Matrix Semigroups over Semigroupoids

In the remainder of this paper, we extend some of the results of [6] concerning Rees matrix semigroups over semigroups, to Rees matrix semigroups over semigroupoids. We observed above that a Rees matrix semigroup over a semigroupoid is necessarily a subsemigroup of a Rees matrix semigroup over the associated categorical-at-zero semigroup. However, it is in general neither a large subsemigroup nor an ideal, so existing results about subsemigroups [8, 9, 17] cannot be used to determine its combinatorial properties. Hence, we construct direct proofs.

The proofs follow a similar pattern to those in [6], but the nature of semigroupoids introduces some extra subtleties. At a technical level, the partial multiplication in semigroupoids requires that finite graphs and paths substitute for alphabets and words respectively. Often, extra care must be taken to consider the definedness or otherwise of products.

Another difference concerns the relationship between Rees matrix semigroups with and without zero. In the case of semigroups, a Rees matrix semigroup with zero over S can be viewed as a construction over S in its own right, but also as a Rees quotient of a Rees matrix semigroup (without zero) over S^0. In [6], the approach is first to prove each result for Rees matrix semigroups (without zero). These results are then used, in conjunction with standard results about combinatorial properties of Rees quotients, to extend to Rees matrix semigroups with zero. Adjoining a zero (or rather, zeros) to a semigroupoid is not quite so straightforward. Hence, we choose to construct proofs for the case of Rees matrix semigroups with zero. The simpler case of Rees matrix semigroups (without zero) then arises as a corollary. The need explicitly to consider a zero element introduces some additional complications.

The following proposition gives a necessary condition for a semigroupoid to be finitely generated. It corresponds to the observation in [6] that, if a Rees matrix semigroup is finitely generated, the ideal generated by the entries of the sandwich matrix is cofinite in S.

Proposition 4. *Suppose $M = M^0(S; F(I), G(\Lambda); P)$ is a finitely generated Rees matrix semigroup with zero over a semigroupoid S. Let*

$$U = SP'S \subseteq S$$

where $P' \subseteq S$ is the set of non-zero entries in the sandwich matrix P. Then U is a subsemigroupoid of S and $S \setminus U$ is finite.

Proof. It is an easy exercise to verify that U is closed under the multiplication of S, and hence is a subsemigroupoid of S.

Now suppose $s \in S \setminus U$. Since $F : I \to S_0$ and $G : \Lambda \to S_0$ are surjective, we can find $i \in I$ and $\lambda \in \Lambda$ such that $\omega(s) = F(i)$ and $\alpha(s) = G(\lambda)$. But then $(i, s, \lambda) \in M$ is clearly indecomposable in M, so (i, s, λ) must exist in any choice of generators for M. If $S \setminus U$ were infinite, it would follow that any choice of generators for M must be infinite, contradicting the assumption that M is finitely generated. □

Theorem 5. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a non-degenerate semigroupoid S, and let U be as defined in the statement of Proposition 4. Then M is finitely generated if and only if*

(i) *the indexing sets I and Λ are finite;*
(ii) *the semigroupoid S is finitely generated; and*
(iii) *$S \setminus U$ is finite.*

The proof is inspired by Section 2 of [6], as discussed above.

Proof. First, suppose $X \subseteq M$ is a finite set of generators for M, and let $i \in I$. We know that $G : \Lambda \to S_0$ is surjective, so there must be some $\lambda \in \Lambda$ satisfying $G(\lambda) = F(i)$. Also, S is non-degenerate, so there must be some $s \in S$ with $\alpha(s) = \omega(s) = F(i)$ so that M has an element (i, s, λ). By assumption, (i, s, λ) can be written as a product of elements of X. But clearly, the first element of this product must be of the form (i, t, μ). Thus, X contains at least as many elements as I. A similar argument shows that X contains at least as many elements as Λ, and it follows that I and Λ are finite. Thus, condition (i) holds.[1]

Let

$$Y = \{s \in S \mid (i, s, \lambda) \in X \text{ for some } i \in I, \lambda \in \Lambda\} \cup \{P_{\lambda i} \mid \lambda \in \Lambda, i \in I, P_{\lambda i} \neq 0\}.$$

Clearly, Y is finite. Now for any $s \in S$, since $F : I \to S_0$ and $G : \Lambda \to S_0$ are surjective, we can find $i \in I$ and $\lambda \in \Lambda$ such that $\alpha(s) = F(i)$ and $\omega(s) = G(\lambda)$. But then $(i, s, \lambda) \in M$, and M is generated by X, so we have

$$(i, s, \lambda) = (i, s_1, \lambda_1)(i_2, s_2, \lambda_2) \dots (i_n, s_n, \lambda)$$

for some $s_1, \dots, s_n \in Y$, $i_2, \dots, i_n \in I$ and $\lambda_1, \dots, \lambda_{n-1} \in \Lambda$. Furthermore, because $\{0\}$ is an ideal of M and $(i, s, \lambda) \neq 0$, we must have $P_{\lambda_m i_{m+1}} \neq 0$ for each m with $1 \leq m < n$, so that

$$(i, s, \lambda) = (i, s_1 P_{\lambda_1 i_2} s_2 P_{\lambda_2 i_3} \dots P_{\lambda_{n-1} i_n} s_n, \lambda).$$

But now we can write

$$s = s_1 P_{\lambda_1 i_2} s_2 P_{\lambda_2 i_3} \dots P_{\lambda_{n-1} i_n} s_n$$

as a product of elements of Y. Thus, Y generates S, and condition (ii) holds.

Condition (iii) follows directly from Proposition 4.

For the converse, assume that conditions (i), (ii) and (iii) hold. By condition (iii) and Theorem 2, we can choose a finite set Y of semigroupoid generators for

[1] We observe in passing that the simpler argument used in the proof of [6, Proposition 2.1] is unsound. If, in their notation, $S \setminus U$ is empty, then $I \times S \setminus U \times J$ is finite (indeed, empty) regardless of whether I and J are finite. Their result stands, since an argument akin to ours can easily be substituted.

the subsemigroupoid U. By the definition of U, we can write each $y \in Y \subseteq U$ in the form

$$y = s(y)P_{\lambda(y)j(y)}s'(y)$$

for some $s(y), s'(y) \in S$, $\lambda(y) \in \Lambda$ and $j(y) \in I$. Now define a set $H \subseteq S$ by

$$\begin{aligned} H &= \{s(y) \mid y \in Y\} \\ &\cup \{s'(y) \mid y \in Y\} \\ &\cup \{s'(y)s(z) \mid y, z \in Y, \alpha(y) = \omega(z)\} \\ &\cup (S \setminus U) \end{aligned}$$

and let

$$X = \{0\} \cup ((I \times H \times \Lambda) \cap M).$$

Clearly, H is finite, and we know I and Λ are finite, so it follows that X is finite. We claim that X generates M. Certainly X generates the zero element $0 \in M$. Now consider a non-zero element $(i, s, \lambda) \in M$. If $s \notin U$ then we have $(i, s, \lambda) \in X$ and we are done.

If $s \in U$, then $s = y_1 y_2 \ldots y_n$ for some $y_1, \ldots, y_n \in Y$. Now we can write

$$\begin{aligned} (i, s, \lambda) &= (i, y_1 y_2 \ldots y_n, \lambda) \\ &= (i, s(y_1)P_{\lambda(y_1)j(y_1)}s'(y_1)s(y_2) \ldots s'(y_n), \lambda) \\ &= (i, s(y_1), \lambda(y_1))(j(y_1), s'(y_1)s(y_2), \lambda(y_2)) \ldots (j(y_n), s'(y_n), \lambda) \end{aligned}$$

as a product of elements of X, as required. $\square$

5. Finitely presented Rees Matrix Semigroups over Semigroupoids

In this section, we find necessary conditions for a Rees matrix semigroup over a semigroupoid to be finitely presentable. In Section 6, we shall see that these conditions are, in fact, necessary and sufficient.

Theorem 6. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a non-degenerate semigroupoid S, and let U be as defined in the statement of Proposition 4. If M is finitely presentable then*

(i) *the indexing sets I and Λ are finite;*
(ii) *the semigroupoid S is finitely presentable; and*
(iii) *$S \setminus U$ is finite.*

The proof is inspired by Section 3 of [6], as discussed above in Section 4 of this paper.

Proof. Suppose M has a finite presentation $\langle A \mid R \rangle$. By Theorem 5, we know that conditions (i) and (iii) hold, so we need only to show that condition (ii) holds.

Let $X \subseteq M$ be the collection of elements represented by single letters in A, and let

$$Y = \{s \in S \mid (i, s, \lambda) \in X \text{ for some } i \in I, \lambda \in \Lambda\}.$$

In fact, by Theorem 3, we can assume without loss of generality that

$$X = \{(i, s, \lambda) \mid s \in Y, F(i) = \omega(s), \alpha(s) = G(\lambda)\}$$

and that each member (i, s, λ) of X is represented by a unique letter $a_{is\lambda}$ in A.

From the proof of Theorem 5, we know that

$$Y \cup \{P_{\lambda i} \mid \lambda \in \Lambda, i \in I, P_{\lambda i} \neq 0\}$$

is a finite choice of generators for S. Let C be a graph with vertex set S_0 and edge set

$$\{c_y \mid y \in Y\} \cup \{d_{\lambda i} \mid \lambda \in \Lambda, i \in I, P_{\lambda i} \neq 0\}$$

where $\alpha(c_y) = \alpha(y), \omega(c_y) = \omega(y)$, $\alpha(d_{\lambda i}) = \alpha(P_{\lambda i})$ and $\omega(d_{\lambda i}) = \omega(P_{\lambda i})$. Then there is a natural surjective homomorphism from C^+ onto S, which takes each c_y to y and each $d_{\lambda i}$ to $P_{\lambda i}$. Clearly, C is a finite graph. We seek a set R' of relations between paths in C such that $C^+/(R')^\#$ is isomorphic to S.

Let $L \subseteq A^+$ be the set of words in A^+ which do not represent 0. We define mappings $\phi : L \to C^+$, $\mu : L \to I$ and $\nu : L \to \Lambda$ by

$$\phi(a_{i_1 y_1 \lambda_1} \ldots a_{i_n y_n \lambda_n}) = c_{y_1} d_{\lambda_1 i_2} c_{y_2} d_{\lambda_2 i_3} \ldots c_{y_n},$$

$$\mu(a_{i_1 y_1 \lambda_1} \ldots a_{i_n y_n \lambda_n}) = i_1$$

and

$$\nu(a_{i_1 y_1 \lambda_1} \ldots a_{i_n y_n \lambda_n}) = \lambda_n.$$

Clearly, $w \in L$ represents $(i, x, \lambda) \in M$ exactly if $\mu(w) = i$, $\nu(w) = \lambda$ and $\phi(w) \in C^+$ represents $x \in S$.

Let $x \in S$ be arbitrary. Since F and G are surjective, we can choose $i \in I$ and $\lambda \in \Lambda$ with $F(i) = \omega(x)$ and $G(\lambda) = \alpha(x)$, so that $(i, x, \lambda) \in M$. Then, since M is generated by X, we can choose some word $w \in L$ representing (i, x, λ). But now $\phi(w) \in C^+$ represents $x \in S$. We have shown that every arrow of S is represented by a path in $\phi(L)$.

Now for each $w \in C^+$ of length no more than 2, choose some path $\chi(w)$ in $\phi(L)$ representing the same element as w. We define a set $R' \subseteq C^+ \times C^+$ of relations by:

$$\begin{aligned} R' &= \{(w, \chi(w)) \mid w \in C^+ \text{ has length no more than } 2\} && (4)\\ &\cup \{(\phi(u), \phi(v)) \mid (u, v) \in R\}. && (5) \end{aligned}$$

We claim that $\langle C \mid R' \rangle$ is a presentation for S, with the obvious isomorphism. It follows easily from the definitions of χ and ϕ that all relations in R' are satisfied in S. It remains to show that every relation satisfied in S is a consequence of the relations in R'. Our strategy is to show first that every path in C^+ is related as a consequence of the relations in R' to a path in $\phi(L)$, and secondly that any relation between paths in $\phi(L)$ which holds in M is a consequence of relations in R'.

Suppose $w = w_1 w_2 \ldots w_n \in C^+$. If $n \leq 2$ then $(w, \chi(w)) \in R'$ and $\chi(w) \in \phi(L)$. Now assume for induction that $n > 2$ and that every w of length no more than $n - 1$ is related by a relation in $(R')^\#$ to a path in $\phi(L)$. Then $w_2 \ldots w_n (R')^\# w'$ for some $w' \in \phi(L)$, say

$$w' = \phi(a_{i_1 x_1 \lambda_1} \ldots a_{i_m x_m \lambda_m}) = c_{x_1} d_{\lambda_1 i_2} c_{x_2} \ldots d_{\lambda_{m-1} i_m} c_{x_m} \in C^+.$$

Now certainly, $\chi(w_1 c_{x_1}) \in \phi(L)$, say

$$\chi(w_1 c_{x_1}) = \phi(a_{j_1 y_1 \mu_1} \ldots a_{j_n y_n \mu_n}).$$

Then, by the definition of ϕ we have

$$\phi(a_{j_1 y_1 \mu_1} \dots a_{j_n y_n \lambda_1} a_{i_2 x_2 \lambda_2} \dots a_{i_m x_m \lambda_m}) = c_{y_1} d_{\mu_1 j_2} \dots c_{y_n} d_{\lambda_1 i_2} \dots d_{\lambda_{m-1} i_m} c_{x_m}.$$

But now

$$\begin{array}{rcl} w & (R')^{\#} & w_1 w' \\ & = & w_1 c_{x_1} d_{\lambda_1 i_2} c_{x_2} \dots d_{\lambda_{m-1} i_m} c_{x_m} \\ & (R')^{\#} & \chi(w_1 c_{x_1}) d_{\lambda_1 i_2} c_{x_2} \dots d_{\lambda_{m-1} i_m} c_{x_m} \\ & = & c_{y_1} d_{\mu_1 j_2} \dots c_{y_n} d_{\lambda_1 i_2} \dots d_{\lambda_{m-1} i_m} c_{x_m} \\ & \in & \phi(L). \end{array}$$

We have shown that every path in C^+ is related by $(R')^{\#}$ to a path in $\phi(L)$.

Now suppose $y, y' \in L$ are such that the relation $(\phi(y), \phi(y'))$ is satisfied in S, that is, such that $\phi(y)$ and $\phi(y')$ represent the same arrow in S. Suppose

$$y = a_{i y_1 \lambda_1} a_{i_2 y_2 \lambda_2} \dots a_{i_n y_m \lambda} \in A^+$$

and

$$y' = a_{i' y'_1 \lambda'_1} a_{i'_2 y'_2 \lambda'_2} \dots a_{i'_n y'_n \lambda'} \in A^+.$$

Let

$$y'' = a_{i y'_1 \lambda'_1} a_{i'_2 y'_2 \lambda'_2} \dots a_{i'_n y'_n \lambda} \in A^+,$$

observing that $\phi(y') = \phi(y'')$. Then $\phi(y)$ and $\phi(y') = \phi(y'')$ represent the same arrow in S, and furthermore, $\mu(y) = \mu(y'')$ and $\nu(y) = \nu(y'')$. By our observation above, we deduce that y and y'' represent the same element of M.

Now since $< A \mid R >$ is a presentation for M, we know that y and y'' are related as a consequence of R. Suppose first that they are related by a single application of a relation in R, that is, that $y = sut$ and $y'' = svt$ for some $s, t \in A^*$ and some relation $(u, v) \in R$. Then from the definition of ϕ, we have

$$\phi(y) = \phi(s) d_{\nu(s)\mu(u)} \phi(u) d_{\nu(u)\mu(t)} \phi(t)$$

and similarly

$$\phi(y') = \phi(y'') = \phi(s) d_{\nu(s)\mu(v)} \phi(v) d_{\nu(v)\mu(t)} \phi(t).$$

But since the relation (u, v) holds in M, we have $\mu(u) = \mu(v)$ and $\nu(u) = \nu(v)$, so that $d_{\nu(s)\mu(u)} = d_{\nu(s)\mu(v)}$ and $d_{\nu(u)\mu(t)} = d_{\nu(v)\mu(t)}$. It follows immediately that $\phi(y)$ and $\phi(y')$ are related by an application of a relation from (5).

Now suppose $y = y_1, y_2, y_3, \dots, y_n = y' \in L$ are such that each y_i is related to y_{i+1} by an application of a relation in R. Then each $\phi(y_i)$ is related to $\phi(y_{i+1})$ by an application of a relation from (5), so $\phi(y)$ is related as a consequence of R to $\phi(y')$. □

6. Rees Matrix Semigroups over Finitely Presented Semigroupoids

We now state and prove the converse of Theorem 6 in the case where the semigroupoid S is a category. We shall then generalise this result to a full converse for Theorem 6.

Theorem 7. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a category S, and let U be as defined in the statement of Proposition 4. If*

(i) *the indexing sets I and Λ are finite;*
(ii) *the category S is finitely presentable as a semigroupoid; and*
(iii) *$S \setminus U$ is finite*

then M is finitely presentable

The proof is inspired by Section 4 of [6], as discussed above in Section 4 of this paper.

Proof. Suppose conditions (i), (ii) and (iii) hold. By (ii), (iii) and Theorem 2, we deduce that U is a finitely presentable subsemigroupoid of S. Let Z be any finite set of generators for U. By the definition of U, we can write each $x \in Z \subseteq U$ as

$$x = s(x)P_{\lambda(x)j(x)}s'(x)$$

for some $s(x), s'(x) \in S$, $\lambda(x) \in \Lambda$ and $j(x) \in I$. Let

$$H = \{s(x), s'(x) \mid x \in Z\} \cup \{1_e \mid e \in S_0\}.$$

Let C be a graph with vertex set S_0 and edge set

$$\{c_{h\lambda ih'} \mid h, h' \in H, i \in I, \lambda \in \Lambda, \alpha(h) = G(\lambda), F(i) = \omega(h')\}$$

where $\alpha(c_{h\lambda ih'}) = \alpha(h')$ and $\omega(c_{hi\lambda h'}) = \omega(h)$. There is a natural homomorphism from C^+ to U which takes $c_{h\lambda ih'}$ to $hP_{\lambda i}h'$. Clearly, Z is contained in the image of this homomorphism, so the homomorphism is surjective. It follows by Theorem 3 that we can find a presentation $\langle C \mid R\rangle$ for U.

We define alphabets A and B by

$$A = \{a_{ih'h\lambda} \mid i \in I, h', h \in H, \lambda \in \Lambda, F(i) = \omega(h'), \alpha(h') = \omega(h), \alpha(h) = G(\lambda)\}$$

and

$$B = \{b_{is\lambda} \mid i \in I, \lambda \in \Lambda, s \in S \setminus U, F(i) = \omega(s), \alpha(s) = G(\lambda)\},$$

observing that A and B are finite. Let z be a new symbol not in A or B, which will represent the zero element of M. There is a natural homomorphism from $(A \cup B \cup \{z\})^+$ to M taking each $a_{ihh'\lambda}$ to (i, hh', λ), each $b_{is\lambda}$ to (i, s, λ) and z to 0. The image of this homomorphism clearly contains the set X of generators for M, found in the proof of Theorem 5, so the homomorphism is surjective.

Let

$$\begin{aligned} L = \{(i, h', w, h, \lambda) \in I \times H \times C^+ \times H \times \Lambda \mid & F(i) = \omega(h'), \alpha(h') = \omega(w), \\ & \alpha(w) = \omega(h), \alpha(h) = G(\lambda)\} \end{aligned}$$

and define a function $\phi : L \to A^+$ by

$$\phi(i, h', c_{h_1\lambda_1 i_1 h'_1} \dots c_{h_m\lambda_m i_m h'_m}, h, \lambda) = a_{ih'h_1\lambda_1}a_{i_1h'_1h_2\lambda_2} \dots a_{i_mh'_mh\lambda}.$$

Let

$$W = \{\phi(i, 1_e, w, 1_f, \lambda) \mid F(i) = e = \omega(w), \alpha(w) = f = G(\lambda)\} \subseteq \phi(L).$$

We observe that

$$\phi(i, h', w_1c_{g\mu jg'}w_2, h, \lambda) = \phi(i, h', w_1, g, \mu)\phi(j, g', w_2, h, \lambda)$$

for all $i, j \in I$, $\lambda, \mu \in \Lambda$ and $g, g', h, h' \in H$, and in particular that the language W is closed under concatenation. Clearly, $\phi(i, h', w, h, j)$, where defined, is a word representing $(i, h\overline{w}h', \lambda) \in M$, where $\overline{w}$ is the arrow of S represented by

$w \in C^+$. In particular, words in W represent elements of the form (i, s, λ) where $s \in U$, that is, non-zero decomposable elements of M.

Let $x \in M$ be arbitrary. If $x = 0$ then x is represented by the letter z. Now suppose $x = (i, s, \lambda)$. If $s \in S \setminus U$ then x is represented by $b_{is\lambda} \in B$. Otherwise, $s \in U$ and U is generated by C, so there exists a path $w \in C^+$ representing s. But now the word $\phi(i, 1_{\omega(w)}, w, 1_{\alpha(w)}, \lambda) \in W$ represents x. We have shown that every element $x \in M$ is represented by a word in $W \cup B \cup \{z\}$

Now for each non-empty word $w \in (A \cup B \cup \{z\})^+$ of length no more than 3, let $\chi(w)$ be a word in $W \cup B \cup \{z\}$ representing the same element of M as w. We observe that, for w of length 2 or 3, the element represented by w is a product of the elements represented by its letters, and so is decomposable in M. It follows easily that $\chi(w) \notin B$ for w of length 2 or 3.[2]

We now define a set $R' \subseteq (A \cup B \cup \{z\})^+ \times (A \cup B \cup \{z\})^+$ of relations by:

$$\begin{aligned} R' &= \{(w, \chi(w)) \mid w \in (A \cup B \cup \{z\})^+ \text{ has length no more than } 3\} && (6)\\ &\cup \{(zx, z), (xz, z) \mid x \in A \cup B \cup \{z\}\} && (7)\\ &\cup \{(\phi(i, h', u, h, \lambda), \phi(i, h', v, h, \lambda)) \\ &\qquad\qquad \mid (i, h', u, h, \lambda), (i, h', v, h, \lambda) \in L, (u, v) \in R\}. && (8)\end{aligned}$$

We claim that $\langle A \cup B \cup \{z\} \mid R' \rangle$ is a presentation for M, with the obvious isomorphism. Clearly, $A \cup B \cup \{z\}$ and R' are both finite. Also, it follows easily from the way we have defined the relations in R' that all are satisfied in M. It remains to show that every relation satisfied in M is a consequence of the relations in R'. Our strategy is to show first that every word in $w \in (A \cup B \cup \{z\})^+$ is related by $(R')^\#$ to some element of $W \cup B \cup \{z\}$, and then that every relation between words in $W \cup B \cup \{z\}$ which is satisfied in M, is a consequence of R'. It then follows by the transitivity of $(R')^\#$ that if a relation (w, w') is satisfied in M then w and w' are related to elements of $W \cup B \cup \{z\}$, and thence to each other.

Suppose $w = w_1 w_2 \ldots w_n \in (A \cup B \cup \{z\})^+$. If $n \leq 3$ then $(w, \chi(w)) \in R'$ and $\chi(w) \in W \cup B \cup \{z\}$, so we are done. Now assume for induction that $n > 3$ and every w of length no more than $n - 1$ is related by a relation in $(R')^\#$ to a word in $W \cup B \cup \{z\}$. Let $w', w'' \in W \cup B \cup \{z\}$ be such that $w_1 w_2 (R')^\# w'$ and $w_3 \ldots w_n (R')^\# w''$. Clearly, the elements represented by $w_1 w_2$ and $w_3 \ldots w_n$ are decomposable, so we have $w', w'' \notin B$. If $w' = z$ or $w'' = z$ then, by repeated application of relations from (7) we have $w (R')^\# w' w'' (R')^\# z$ and we are done. Otherwise, we have $w', w'' \in W$. But W is closed under concatenation, so we have $w (R')^\# w' w''$ where $w' w'' \in W$, as required. We have shown that every word is related by an element of $(R')^\#$ to a word in $W \cup B \cup \{z\}$.

Now suppose $w, w' \in W \cup B \cup \{z\}$ are such that the relation (w, w') is satisfied in M, that is, such that w and w' represent the same element of M. We have

[2]In [6, Lemma 4.1], an equivalent set of relations is constructed by quantifying over the index sets I and J and over H rather than over the alphabets A and B. In a sense, this duplicates work already done in defining the alphabets. Our approach simplifies the proof somewhat, at the price of introducing some extra redundant relations. In particular, one can show that the relations $(w, \chi(w))$ are unnecessary for all w of length 3 and for $w = a_1 a_2$ of length 2 with $a_1, a_2 \in A$.

already observed that words in W represent non-zero decomposable words in M, and it follows easily that the letters in $B \cup \{z\}$ are unique representatives in $W \cup B \cup \{z\}$ for their respective elements. Thus, if w or w' is in $B \cup \{z\}$ then we must have $w = w'$, so we are done. Otherwise, we have $w, w' \in W$.

Now by the definition of W, we have $w = \phi(i, 1_{\omega(x)}, x, 1_{\alpha(x)}, \lambda)$ and $w' = \phi(i', 1_{\omega(x')}, x', 1_{\alpha(x')}, \lambda')$ for some $(i, 1_{\omega(x)}, x, 1_{\alpha(x)}, \lambda), (i', 1_{\omega(x')}, x', 1_{\alpha(x')}, \lambda') \in L$. In order for w and w' to represent the same element in M, it must be that $i = i'$, that $\lambda = \lambda'$ and that x and x' represent the same arrow in S. Now $\langle C \mid R \rangle$ is a presentation for U, so we must have that x and x' represent an element in U, and are related as a consequence of R.

Suppose first that they are related by a single application of a relation in R, that is, that $x = sut$ and $x' = svt$ for some $s, t \in C^*$ and some relation $(u, v) \in R$. If s and t are non-empty then we can write $s = s'c_{g\mu jg'}$ and $t = c_{h\nu kh'}t'$ for some $c_{g\mu jg'}, c_{h\nu kh'} \in C$ and $s, t \in C^*$, and now we have:

$$\begin{aligned}
w &= \phi(i, 1_{\omega(x)}, x, 1_{\alpha(x)}, \lambda) \\
&= \phi(i, 1_{\omega(x)}, sut, 1_{\alpha(x)}, \lambda) \\
&= \phi(i, 1_{\omega(x)}, s'c_{g\mu jg'}uc_{h\nu kh'}t', 1_{\alpha(x')}, \lambda) \\
&= \phi(i, 1_{\omega(x)}, s', g, \mu)\phi(j, g', u, h, \nu)\phi(k, h', t', 1_{\alpha(x)}, \lambda)
\end{aligned}$$

and similarly

$$w' = \phi(i, 1_{\omega(x)}, s', g, \mu)\phi(j, g', v, h, \nu)\phi(k, h', t', 1_{\alpha(x)}, \lambda).$$

It follows that w and w' are related by a relation from (8). Similar arguments apply in the cases where either or both of s and t are empty. An easy inductive argument, akin to that concluding the proof of Theorem 6, now extends this to the case in which x and x' are related by a chain of applications of relations in R. □

We are now in a position to state the main result of this section.

Theorem 8. *Let $M = M^0(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup with zero over a non-degenerate semigroupoid S, and let U be as defined in the statement of Proposition 4. Then M is finitely presentable if and only if*

(i) *the indexing sets I and Λ are finite;*
(ii) *the semigroupoid S is finitely presentable; and*
(iii) *$S \setminus U$ is finite.*

Proof. Suppose (i), (ii) and (iii) hold. Let S' be the category obtained from S by adjoining an identity arrow at each object. Since I is finite and $F : I \to S_0$ is surjective, S has only finitely many objects. It follows that S is a cofinite subsemigroupoid of S' so, by Theorem 2, S' is finitely presentable.

Now let $M' = M^0(S', F(I), G(\Lambda); P)$. It is easily verified that this construction satisfies the conditions of Theorem 7, so M' is finitely presentable. Furthermore, we have

$$M = M' \setminus \{(i, 1_e, \lambda) \mid F(i) = e = G(\lambda) \in S_0\}$$

so that M is a cofinite subsemigroup in M'. By [17, Theorem 1.3] it follows that M is finitely presentable.

The converse implication is given by Theorem 6. $\square$

7. Rees Matrix Semigroups Without Zero

Our results about Rees matrix semigroups with zero adapt easily to the case of Rees matrix semigroups without zero.

Theorem 9. *Let $M = M(S; F(I), G(\Lambda); P)$ be a Rees matrix semigroup (without zero) over a (strongly connected) semigroupoid S, and let U be as defined in the statement of Proposition 4. Then M is finitely generated [finitely presentable] if and only if*

(i) *the indexing sets I and Λ are finite;*
(ii) *the semigroupoid S is finitely generated [respectively, finitely presentable]; and*
(iii) *$S \setminus U$ is finite.*

Proof. By Theorem 2, the semigroup $M = M^0(S; F(I), G(\Lambda); P) \setminus \{0\}$ is finitely generated [finitely presentable] if and only if the Rees matrix semigroup with zero $M^0(S; F(I), G(\Lambda); P)$ is finitely generated [finitely presentable]. By Theorem 5 [Theorem 8], this is so if and only if conditions (i), (ii) and (iii) hold. $\square$

A combination of Theorems 5, 8 and 9 yields the following additional implication.

Corollary 10. *Suppose M is a finitely generated Rees matrix semigroup (with or without zero) over a non-degenerate finitely presentable semigroupoid S. Then M is finitely presentable.*

Acknowledgements

The author would like to thank John Fountain for all his advice and guidance, Vicky Gould for her comments on the draft, and Kirsty for her encouragement. This work was supported by an EPSRC Doctoral Studentship.

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