MATH10202 Linear Algebra A 2019-20

Exercise Sheet 0 (with solutions)

These exercises are to do **in the Week 1 Feedback Supervision**. They revise material from **MATH10101 Foundations of Pure Mathematics** which will be needed again this term, so **you may wish to take your MATH10101 notes to the supervision**.

0.1. Let $f : A \to B$ be a function. What does it mean for f to be (a) injective, (b) surjective and (c) bijective?

Solution to 0.1. *If you can't remember this, dig out your MATH10101 notes and look it up — it is very important!*

0.2. Decide whether each of the following functions is (a) injective, (b) surjective or (c) bijective:

(i) $g : \mathbb{C} \to \mathbb{C},\ g(x) = x^2$;

(ii) $k : \{1, 2\} \to \{1, 2\},\quad k(x) = 3 - x$.

Solution to 0.2.

(i) g is surjective (since every complex number has a complex square root) but not injective (since for example $g(1) = g(-1) = 1$*) and hence not bijective.*

(ii) k is injective and surjective, and hence also bijective. (In other words, it is a permutation of the set $\{1, 2\}$*, namely the transposition* (12)*.)*

0.3. Consider the function:

$$h \ :\ \mathbb{R}^2 \to \mathbb{R}^2, \quad h(x, y) = \left(\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y,\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \right)$$

Try applying h to some pairs (x, y). If we think of each pair (x, y) as the coordinates of a point in the plane, can you describe geometrically the action of the function h? Do you think h is injective, surjective and/or bijective?

Solution to 0.3. *For example, try applying it to* $(1, 0)$, $(0, 1)$, $(0, 0)$, $(2, 0)$, $(2, 2)$, $(-3, 4)$, $(-1, -1)$. *Can you see yet what is happening? If we think of* (x, y) *as a point in the plane,* $h(x, y)$ *is the point we get by rotating it through* $45°$ *anticlockwise around the origin. Given that this is what it does, it should be clear that it is a bijective function, for example because it has an obvious inverse (the rotation through* $45°$ ***clockwise*** *around the origin).*

0.4. If A is a set, what does $|A|$ denote?

Solution to 0.4. $|A|$ *is the cardinality of the set* A*, that is, the number of elements of* A*, which could be a non-negative integer or something infinite.*

0.5. Let $f : A \to B$ be a function. Mark each of the following statements true or false:

(i) If f is bijective then $|A| = |B|$;

(ii) If $|A| = |B|$ then f is bijective;

(iii) If f is surjective then $|A| < |B|$;

(iv) If $|A| > |B|$ then f is not injective;

(v) If $|A| = 1$ then f is injective.

(vi) f is surjective if and only if $\mathrm{Im} f = B$.

(vii) All flying pigs carry umbrellas.

(viii) Every bijective function has an inverse.

For those which are false, find a counterexample.

Solution to 0.5.

(i) True.

(ii) False, a counterexample being $f : \mathbb{Z} \to \mathbb{Z}, x \mapsto x^2$.

*(iii) So completely false that **every** surjective function is a counterexample! What **is** true is more or less the opposite: if* f *is surjective then* $|A| \geq |B|$.

*(iv) True, and important enough to have its own name: the **pigeon-hole principle**. (If you assign pigeons to holes when you have more pigeons than holes, then at least two pigeons will have to share.)*

(v) True. (If $x, y \in A$ *are such that* $f(x) = f(y)$ *then we must have* $x = y$ *simply because* $|A| = 1$.*)*

(vi) True, as a simple consequence of the definitions.

*(vii) True. If you don't believe me, show me a flying pig **without** an umbrella! More formally we might phrase this as "*X *is a flying pig* $\implies$ X *is carrying an umbrella", and remember that* $P \implies Q$ *is true whenever* P *is false! (Implications which are true because the left-hand-side is always false are called **vacuously true**.)*

(viii) True (and very important!).

0.6. Suppose we have two straight lines in the plane. Let $S \subseteq \mathbb{R}^2$ be the set of points on one line, and $T \subseteq \mathbb{R}^2$ the set of points on the other. What are the possible values of $|S \cap T|$?

Solution to 0.6. *Recall that* $S \cap T$ *is the set of points which lie in both* S *and* T*, which in this case is the set of points which lie on both lines. As long as the lines aren't parallel, they'll meet in a single point so we have* $|S \cap T| = 1$*. If they are parallel but different they don't meet at all, so* $|S \cap T| = 0$*. Finally, they could actually be **the same line**, in which case* $S \cap T = S = T$ *is the whole line and* $|S \cap T| = c$ *(recall that* c *is the (infinite) cardinality of* $\mathbb{R}$*.). If you thought of the last possibility then congratulations: you are pedantic enough to be a mathematician! :-)*

0.7. What are the invertible elements of $\mathbb{Z}_6$ (with modulo 6 arithmetic)?

Solution to 0.7. *Recall that the elements of* $\mathbb{Z}_6$ *are the congruence classes modulo* 6*, that is:*

$$[0]_6,\ [1]_6,\ [2]_6,\ [3]_6,\ [4]_6 \text{ and } [5]_6.$$

The invertible ones are those of the form $[n]_6$ *where* n *is coprime to* 6*, so that is just* $[1]_6$ *and* $[5]_6$*.*