Restriction Semigroups

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Where to begin?

Restriction semigroups may be obtained as/from:

- Varieties of algebras
- Representation by (partial) mappings
- Generalised Green's relations
- Inductive categories and constellations

Notation

S will always denote a semigroup

$E(S)$ is the set of idempotents of S and

$E \subseteq E(S)$

The relations $\mathcal{R}$ and $\mathcal{L}$

- For any $a, b \in S$ we have

$$\begin{aligned} a \,\mathcal{R}\, b &\Leftrightarrow aS^1 = bS^1 \\ &\Leftrightarrow \exists s, t \in S^1 \text{ with } a = bs \text{ and } b = at. \end{aligned}$$

- For any $a, b \in S$ we have

$$\begin{aligned} a \,\mathcal{L}\, b &\Leftrightarrow S^1 a = S^1 b \\ &\Leftrightarrow \exists s, t \in S^1 \text{ with } a = sb \text{ and } b = ta. \end{aligned}$$

- $\mathcal{R}$ ($\mathcal{L}$) is a left (right) congruence
- $\mathcal{R}$ and $\mathcal{L}$ are the universal relation on any group

The relations $\mathcal{R}$ and $\mathcal{L}$: regular and inverse semigroups

Definition S is *regular* if for all $a \in S$ there exists $x \in S$ with $a = axa$.

Notice that if $a = axa$, then $ax, xa \in E(S)$ and

$$ax \, \mathcal{R} \, a \, \mathcal{L} \, xa.$$

Fact S is regular if and only if every $\mathcal{R}$-class (or $\mathcal{L}$-class) contains an idempotent.

Definition S is inverse if S is regular and $E(S)$ is a semilattice.

Fact S is inverse if and only every element has a unique inverse, i.e. for all $a \in S$ there exists a unique a' in S such that

$$a = aa'a \text{ and } a' = a'aa'.$$

Fact S is inverse if and only if every $\mathcal{R}$-class and every $\mathcal{L}$-class contains a unique idempotent.

The relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$

- The relation $\widetilde{\mathcal{R}}_E$ is defined by $a \,\widetilde{\mathcal{R}}_E\, b$ if and only if
$$ea = a \Leftrightarrow eb = b$$
for all $e \in E$.
- Note if $a \,\widetilde{\mathcal{R}}_E\, e \in E$, then as $ee = e$ we have $ea = a$.
- The relation $\widetilde{\mathcal{L}}_E$ is defined by $a \,\widetilde{\mathcal{L}}_E\, b$ if and only if
$$ae = a \Leftrightarrow be = b$$
for all $e \in E$.
- $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ are equivalence relations.
- If M is a monoid and $E = \{1\}$, then $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ are universal.
- these relations were introduced by El-Qallali in his 1980 thesis [5] (under Fountain) in case $E = E(S)$, later generalised by Lawson [13]

The relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ - connection to $\mathcal{R}$ and $\mathcal{L}$

Fact For any semigroup S and any E

$$\mathcal{R} \subseteq \widetilde{\mathcal{R}}_E.$$

Proof Let $a \,\mathcal{R}\, b$. Then $a = bs$ and $b = at$ for some $s, t \in S^1$.

Hence

$$ea = a \Rightarrow eat = at \Rightarrow eb = b \Rightarrow ebs = bs \Rightarrow ea = a.$$

Fact If S is regular and $E = E(S)$, then $\widetilde{\mathcal{R}}_E = \mathcal{R}$.

Proof If $a \,\widetilde{\mathcal{R}}_{E(S)}\, b$ and $a = axa, b = byb$, then $b = axb$ and $a = bya$.

Restriction semigroups: first definition

Definition A semigroup S is **left restriction** with **distinguished semilattice** E if:

- E is a semilattice;
- every $\widetilde{\mathcal{R}}_E$-class contains an idempotent of E;
  it is then easy to see that for every $a \in S$ the $\widetilde{\mathcal{R}}_E$-class of a contains a unique element of E, which we call a^+;
- the relation $\widetilde{\mathcal{R}}_E$ is a left congruence and
- the **left ample** condition (AL) holds:

$$\text{for all } a \in S \text{ and } e \in E,\ ae = (ae)^+a.$$

Right restriction semigroups are defined dually. A semigroup is **restriction** if it is left and right restriction *with respect to the same distinguished semilattice*.

Example Let M be a monoid. Then M is restriction with distinguished semilattice $E = \{1\}$.

Inverse semigroups are restriction

Let S be an inverse semigroup. Then with $E = E(S)$:

- E is a semilattice;
- $\widetilde{\mathcal{R}}_E = \mathcal{R}$ is a left congruence;
- every $\mathcal{R}$-class contains an idempotent: we have

$$a^+ = aa';$$

- for any $a \in S$ and $e \in E$

$$(ae)^+a = (ae)(ae)'a = ae(ea')a = ae(a'a) = a(a'a)e = ae.$$

Hence S is left restriction (w.r.t. $E(S)$); dually S is right restriction, so that S is restriction.

Semigroups and representations

- Every semigroup S embeds in a **full transformation semigroup** $\mathcal{T}_X$
- Every group embeds in a **symmetric group** $\mathcal{S}_X$
- Every inverse semigroup S embeds (as an inverse semigroup) in the **symmetric inverse semigroup** $\mathcal{I}_X$

Restriction semigroups: representations

$\mathcal{T}_X, \mathcal{S}_X$ and $\mathcal{I}_X$ are all subsemigroups of the semigroup $\mathcal{PT}_X$ of all partial mappings of X.

- $\mathcal{PT}_X$ is left restriction with distinguished semilattice

  $$E = \{I_Y : Y \subseteq X\}$$

  and with

  $$\alpha^+ = I_{\mathrm{dom}\ \alpha}.$$

- S is left restriction if and only if it embeds in some $\mathcal{PT}_X$ in a way that preserves $^+$ (folklore: Trokhimenko [21]).

Restriction semigroups: varieties

Let $S = (S, \cdot, ^+)$ be a semigroup equipped with a unary operation $^+$ (*that is, S is a* **unary semigroup**).

Fact S is **left restriction** with distinguished semilattice

$$E = \{a^+ : a \in S\}$$

if and only if the following identities hold:

$$x^+x = x,\ x^+y^+ = y^+x^+,\ (x^+y)^+ = x^+y^+,\ xy^+ = (xy)^+x.$$

If the above identities hold then for any $a^+ \in E$,

$$a^+ = (a^+a)^+ = a^+a^+$$

so that we see E is a semilattice.

Restriction semigroups: varieties

- Consequently, left restriction semigroups form a **variety** of unary semigroups.
- Dually, right restriction semigroups form a variety of unary semigroups, with unary operation denoted by *, satisfying the left/right duals of the axioms above.
- A bi-unary semigroup is restriction if and only if satisfies the identities for left and right restriction semigroups together with

$$(a^*)^+ = a^* \text{ and } (a^+)^* = a^+.$$

- Since (left) restriction semigroups form varieties, free objects exist.
- The free (left) restriction semigroup on any set X embeds into the free inverse semigroup on X ([9, 8]).

A bit of history

Different schools arrived at (left) restriction semigroups via different directions from 1960s onwards:

- Schweizer, Sklar, Schein, Trokhimenko: **function systems** [16, 17, 18, 19, 20]
  Let T be a subsemigroup of $\mathcal{PT}_X$ or $\mathcal{B}_X$ (semigroup of binary relations on X).
  T may be equipped with additional operations such as $^+$, $\cap$, $(f, g) \mapsto f^+g$ etc.
  Can such T be axiomatised by first order formulae? By identities?
- Lawson: **Ehresmann semigroups** [13]
  Lawson found a correspondence between Ehresmann semigroups and certain categories equipped with two orderings. As a special case, restriction semigroups correspond to inductive categories.

A bit more history

- Jackson and Stokes: **closure operators** [10]
  Introduced 'twisted C-semigroups', with an axiomatisation equivalent to the one given here.
- Manes, Cockett, Lack: **category theory, computer science** [2, 14]. Gave the axioms above. Also interested in restriction *categories*.
- Fountain: **generalisations of regular and inverse semigroups** [6].
- Jones: **P-restriction semigroups** obtained from *regular $*$-semigroups* [11].

The relations $\mathcal{R}^*$ and $\mathcal{L}^*$

- The relation $\mathcal{R}^*$ on S is defined by the rule that $a\,\mathcal{R}^*\,b$ if and only if

  $$xa = ya \Leftrightarrow xb = yb$$

  for all $x, y \in S^1$.
- $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$
- A monoid M is **left PP** if every principal left ideal is projective.
- M is left PP if and only if every $\mathcal{R}^*$-class contains an idempotent.

This observation by Fountain [6] led to the introduction of **abundant, adequate semigroups, etc.**

(Left) ample semigroups

Definition A semigroup S is **left ample** (formerly, **left type A**) if $E(S)$ is a semilattice, every $\mathcal{R}^*$-class contains an idempotent, and for all $a \in S, e \in E(S)$,

$$ae = (ae)^+ a$$

where a^+ is the unique idempotent in the $\mathcal{R}^*$-class of a.

Equivalently, S is left ample if and only if it is left restriction and $\mathcal{R}^* = \widetilde{\mathcal{R}}_E$.

Right ample semigroups are defined dually, and a semigroup is **ample** if it is both left and right ample.

Fact A unary semigroup is left ample if and only if it embeds in some $\mathcal{I}_X$ [7].

Fact (Left) ample semigroups form a quasi-variety; the variety they generate is the variety of (left) restriction semigroups [9, 8].

Inverse semigroups: groups and semilattices

There are several approaches to structure of inverse semigroups, using groups and semilattices. These may be adapted to (left) restriction semigroups.

McAlister's approach uses **proper covers**: if S is inverse then it has a **proper** preimage $\widehat{S}$ such that $E(\widehat{S}) \cong E(S)$ and such that the structure of $\widehat{S}$ is known - it is isomorphic to a **P-semigroup**.

P-semigroups are closely related to semidirect products.

Restriction semigroups: monoids and semilattices

Let S be left restriction.

- S is **reduced** if $|E| = 1$. A reduced left restriction semigroup is simply a monoid!
- σ_E is the least congruence **identifying all the idempotents of** E.
- The left restriction semigroup S/σ_E is reduced.
- A left restriction semigroup S is **proper** if $\widetilde{\mathcal{R}}_E \cap \sigma_E = \iota$.
- If S is proper left restriction, then $\theta : S \to E \times S/\sigma_E$ given by

$$s\theta = (s^+, s\sigma_E)$$

  is an injection.

Let M be a monoid and Y a set. Then M **acts on the left of** Y if there is a map

$$M \times Y \to Y; \quad (m, y) \mapsto m \cdot y,$$

such that

$$1 \cdot y = y \text{ and } (mn) \cdot y = m \cdot (n \cdot y).$$

Suppose now that Y is a semigroup. Then M **acts by morphisms** if, in addition,

$$m \cdot (yz) = (m \cdot y)(m \cdot z).$$

In this case, define a product on $Y \times M$ by

$$(y, m)(z, n) = (y(m \cdot z), mn).$$

This product is associative, yielding the **semidirect product** $Y * M$.

An example: left restriction semigroups

- If M is a group and Y a semilattice, $Y * M$ is proper inverse.
- If M is a monoid and Y a semilattice, $Y * M$ is proper left restriction.
- Let S be a left restriction monoid with distinguished semilattice E. Define
$$s \cdot e = (se)^+.$$
Then this is an action of S on E.
- Let $s \in S$ and $e, f \in E$. Then from the ample condition $ae = (ae)^+a$ and the identity $(x^+y)^+ = x^+y^+$,
$$s \cdot ef = (sef)^+ = \left((se)^+sf\right)^+ = (se)^+(sf)^+ = (s \cdot e)(s \cdot f).$$
- From the above, $E * S$ is proper left restriction.

Proper covers of left restriction semigroups

Let S be left restriction.

- A **proper cover** of S is a proper left restriction semigroup $\widehat{S}$ and an onto morphism $\theta : \widehat{S} \twoheadrightarrow S$ such that θ separates distinguished idempotents.
- If S is a monoid then

$$\widehat{S} = \{(e, s) : e \leq s^{+}\} \subseteq E * S$$

  is a proper cover of S
- Every left restriction semigroup has a proper cover [1].

Proper left restriction semigroups: a recipe

Let T be a monoid acting on the left of a semilattice $\mathcal{X}$ via morphisms. Suppose that $\mathcal{X}$ has subsemilattice $\mathcal{Y}$ with upper bound ε such that
(a) for all $t \in T$ there exists $e \in \mathcal{Y}$ such that $e \leq t \cdot \varepsilon$
(b) if $e \leq t \cdot \varepsilon$ then for all $f \in \mathcal{Y}$, $e \wedge t \cdot f \in \mathcal{Y}$.
Then $(T, \mathcal{X}, \mathcal{Y})$ is a **strong left M-triple**.

For a strong left M-triple $(T, \mathcal{X}, \mathcal{Y})$ we put

$$\mathcal{M}(T, \mathcal{X}, \mathcal{Y}) = \{(e, t) \in \mathcal{Y} \times T : e \leq t \cdot \varepsilon\}$$

and define

$$(e, s)(f, t) = (e \wedge s \cdot f, st), \ (e, s)^+ = (e, 1).$$

Then $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ is proper left restriction.

The 'covering' approach for left restriction semigroups

Theorem A left restriction semigroup S is proper if and only if it is isomorphic to some $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ [1].

Important point In the above result, we can take

$$T = S/\sigma_E \text{ and } \mathcal{Y} = E.$$

By replacing T with a right cancellative monoid, we can specialise to the left ample case: see also Fountain [6] and Lawson [12].

Proper restriction semigroups

- A restriction semigroup S is *proper* if

$$\widetilde{\mathcal{R}}_E \cap \sigma_E = \iota = \widetilde{\mathcal{L}}_E \cap \sigma_E.$$

- Every restriction semigroup has a proper cover [8]
- If S is proper restriction, then as S is proper left restriction,

$$S \cong \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$$

where $T = S/\sigma_E$ and $\mathcal{Y} = E$, and as S is proper right restriction,

$$S \cong \mathcal{M}'(\mathcal{Y}, \mathcal{X}', T),$$

where $\mathcal{M}'(\mathcal{Y}, \mathcal{X}', T)$ is constructed from T acting on the right of a semilattice $\mathcal{X}'$.

- Clearly the left and right actions of T must be connected in some way.

A structure theorem for proper restriction semigroups: the set-up

Definition Let T be a monoid, acting partially on the left and right of a semilattice $\mathcal{Y}$, via $\cdot$ and $\circ$ respectively. Suppose that both actions preserve the partial order and the domains of each $t \in T$ are order ideals. Suppose in addition that for $e \in \mathcal{Y}$ and $t \in T$, the following and their duals hold:
(a) if $\exists e \circ t$, then $\exists t \cdot (e \circ t)$ and $t \cdot (e \circ t) = e$;
(b) for all $t \in T$, there exists $e \in \mathcal{Y}$ such that $\exists e \circ t$.

Then $(T, \mathcal{Y})$ is a **strong M-pair**.

We put

$$\mathcal{M}(T, \mathcal{Y}) = \{(e, s) \in \mathcal{Y} \times T : \exists e \circ s\}$$

and define operations by

$$(e, s)(f, t) = (s \cdot (e \circ s \wedge f), st),\ (e, s)^+ = (e, 1) \text{ and } (e, s)^* = (e \circ s, 1).$$

A structure theorem for proper restriction semigroups: the result

Theorem: Cornock and G [4] If $(T, \mathcal{Y})$ is a strong M-pair, then

$$\mathcal{M}(T, \mathcal{Y}) \cong \mathcal{M}'(\mathcal{Y}, T),$$

where $\mathcal{M}'(\mathcal{Y}, T)$ is constructed dually to $\mathcal{M}(T, \mathcal{Y})$.

Theorem: Cornock and G [4] A semigroup is proper restriction if and only if it is isomorphic to some $\mathcal{M}(T, \mathcal{Y})$.

Corollary: Lawson [12] A semigroup is proper ample if and only if it is isomorphic to $\mathcal{M}(C, \mathcal{Y})$ for a cancellative monoid C.

Corollary: Petrich and Reilly, [15] A semigroup is proper inverse if and only if it is isomorphic to $\mathcal{M}(G, \mathcal{Y})$ for a group G.

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