Rees monoids, self-similar groups and fractals

Alistair Wallis

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History

- David Rees 1948 studies ideal structure of cancellative monoids
- Perrot 1970's studies inverse hull
- Cohn and von Karger prove rigid monoids embed in groups
- 1980's study of automatic groups
- 1990's study of self-similar groups
- Recently, Alan Cain has studied automaton semigroups

LRMs

Definition

A monoid M is said to be a *left Rees monoid* (LRM) if the following hold:

1. M is left cancellative: $ab = ac \Rightarrow b = c$ for all $a, b, c \in M$
2. Incomparable principal right ideals are disjoint: $aM \subseteq bM$ or $bM \subseteq aM$ or $aM \cap bM = \emptyset$ for all $a, b \in M$
3. Each principal right ideal is properly contained in only a finite number of prinicipal right ideals

We define *right Rees monoids* analagously: right cancellative monoids with disjoint incomparable principal left ideals and finite inclusion of principal left ideals

Group of units

For a monoid M we will denote by $G(M)$ the group of units of M; that is, the elements which are uniquely invertible in the group theoretic sense.

Big Proposition

Proposition

Let M be an LRM. Let X be a transversal of the generators of the maximal proper principal right ideals, and denote by X^* the submonoid generated by the set X. Then the monoid X^* is free, $M = X^*G(M)$ and every element of M can be written uniquely as a product of an element of X^* and an element of $G(M)$.

Self-similar group actions

Definition

Let G be a group and X^* be the free monoid on X. We will say that G and X^* act self-similarly on each other if there exist two maps $G \times X^* \to X^*$, $(g, x) \mapsto g \cdot x$ called the *action* and $G \times X^* \to G$, $(g, x) \mapsto g|_x$ called the *restriction* satisfying the following 8 axioms:

(SS1)	$1 \cdot x = x$	(SS2)	$(gh) \cdot x = g \cdot (h \cdot x)$
(SS3)	$g \cdot 1 = 1$	(SS4)	$g \cdot (xy) = (g \cdot x)(g|_x \cdot y)$
(SS5)	$g|_1 = g$	(SS6)	$g|_{xy} = (g|_x)|_y$
(SS7)	$1|_x = 1$	(SS8)	$(gh)|_x = g|_{(h \cdot x)} h|_x$

for all $x, y \in X^*$ and $g, h \in G$.

Self-similar group actions

Proposition

Let M be an LRM. Then M admits a self-similar action.

Proof.

Let $x \in X^*$ and $g \in G(M)$. Since $M = X^*G(M)$ uniquely, we can write gx uniquely as a product of an element of X^* and one of $G(M)$. So define $gx = g \cdot xg|_x$. It is easy to check that this definition satifies the above axioms. □

Zappa-Szép products

Definition

Let G be a group and X^* be the free monoid on X, such that there is a self-similar action of G on X^*. We will define the *Zappa-Szép* product $X^* \bowtie G$ to be their Cartesian product with the following multiplication:

$$(x, g)(y, h) = (xg \cdot y, g|_y h)$$

for $x, y \in X^*$ and $g, h \in G$.

Zappa-Szép products

Theorem

Every left Rees monoid is isomorphic to a Zappa-Szép product of a free monoid and a group. Conversely every Zappa-Szép product of a free monoid and a group is a left Rees monoid

Remark

What this says is that left Rees monoids and self-similar actions are one and the same thing

Green's $\mathcal{R}$ relation

Definition

Let M be a monoid, $s, t \in M$. Then $s\mathcal{R}t$ if $sM = tM$.

Remark

The relation $\mathcal{R}$ is an equivalence relation (in fact it is a left congruence)

Lemma

Let $M = X^*G$ be an LRM, $x, y \in X^*$, $g, h \in G$. Then $xg\mathcal{R}yh$ if, and only if, $x = y$.

Rees monoids

Lemma

Let M be a left Rees monoid which is also right cancellative. Then M is also a right Rees monoid.

Because of this lemma we will call right cancellative left Rees monoids *Rees monoids*

Restriction map

Definition

For each $x \in X^*$, define $\rho_x : G \to G$ by $g \to g|_x$ and define $\phi_x : G_x \to G$ to be the restriction of ρ_x to G_x.

Lemma

An LRM is right cancellative iff ϕ_x is injective for all $x \in X$

Definition

An LRM with ρ_x bijective for all $x \in X^*$ is called *symmetric*.

Symmetric Rees monoids

Theorem

An LRM M (which is a Zappa-Szép product of a free monoid X^* and a group G) can be extended to the Zappa-Szép product of the free group $FG(X)$ and the group G if, and only if, M is symmetric.

Proof.

($\Rightarrow$) Straightforward: uniqueness and existence of restrictions
($\Leftarrow$) Define $g|_{x^{-1}} := \rho_x^{-1}(g)$ for $x \in X$ and extend the restriction to $g|_x$ for $x \in FG(X)$ by using rule (SS6):
$g|_{x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_n^{\epsilon_n}} = ((g|_{x_1^{\epsilon_1}})|_{x_2^{\epsilon_2}}) \ldots |_{x_n^{\epsilon_n}} \quad x_i \in X, \epsilon_i = \pm 1$. For $x \in X^*$, $g \in G$ define $g \cdot x^{-1} := (g|_{x^{-1}} \cdot x)^{-1}$. $\square$

Monoid HNN-extensions

Definition

Let S be a monoid, T a submonoid of S and let $\alpha : T \to S$ be an injective homomorphism. Then M is a *monoid HNN-extension* of S if M can be defined by the following monoid presentation

$$M = \langle S, t | \mathcal{R}(S), \quad ts = \alpha(s)t \quad \forall s \in T \rangle,$$

where $\mathcal{R}(S)$ denotes the relations of S

Monoid multiple HNN-extensions

Definition

Let S be a monoid, $T_1, \ldots, T_n$ submonoids of S and let $\alpha_i : T_i \to S$ be injective homomorphisms for each i. Then M is a *monoid multiple HNN-extension* of S if M can be defined by the following monoid presentation

$$M = \langle S, t_1, \ldots, t_n | \mathcal{R}(S), \quad t_i s = \alpha_i(s) t_i \quad \forall s \in T_i, i = 1, \ldots, n \rangle,$$

where $\mathcal{R}(S)$ denotes the relations of S

Classification theorem

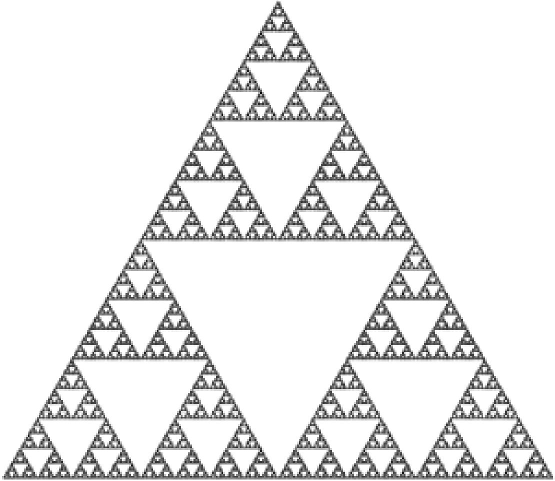
Theorem

Let S be a group, $T_1, \ldots, T_n$ finite index subgroups of S and let $\alpha_i : T_i \to S$ be injective homomorphisms for each i, and let M be the monoid multiple HNN-extension of S as defined above. Then M is a Rees monoid. Furthermore, every Rees monoid can be constructed in this manner

Generalisation to categories

- Left Rees categories
- Self-similar groupoid actions
- Category HNN-extensions

Sierpinski Gasket



Applying the theorems

- M is the monoid of similarity contractions the Sierpinski gasket
- R, L and T be the maps which halve the gasket and translate it, respectively, to the right, left and top of itself
- ρ is rotation by $2\pi/3$ degrees
- σ is reflection in the verticle axis
- Group of isometries:

$$G = \langle \rho, \sigma | \rho^3 = \sigma^2 = 1, \rho\sigma = \sigma\rho^2 \rangle$$

- M is a left Rees monoid, $X = \{L, R, T\}$, G group of units
- $g|_x = g$ for every $g \in G$, $x \in X$, so symmetric Rees monoid
- $G_T = \{1, \sigma\}$
- Monoid presentation of M:

$$M = \langle \rho, \sigma, T | \rho^3 = \sigma^2 = 1, \rho\sigma = \sigma\rho^2, \sigma T = T\sigma \rangle$$

Thank you for listening