An Inverse Monoid Approach to Thompson's Group V and Generalisations

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19 May 2011
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Properties

1. V contains every finite group
2. V is simple
3. V is finitely presented
4. V has type FP_∞
5. V has solvable word problem
6. V has solvable conjugacy problem
7. V has a subgroup isomorphic to $F_2 \times F_2$
8. The generalised word problem for V is undecidable

Right ideals of A^*

- $A = \{a_1, \ldots, a_k\}$; $u, v \in A^*$. u is a prefix of v if $v \in uA^*$.
- Prefix code P over A: $P \subseteq A^*$ and $uA^* \cap vA^* = \emptyset \; \forall u, v \in P$.
- P is maximal if for a prefix code Q over A,
$$P \subseteq Q \Rightarrow P = Q.$$
- If R a right ideal of A^*, then
  (i) $R = PA^*$ for a uniquely determined prefix code P;
  (ii) P is the unique minimal set of generators for R.
- R is essential if $R \cap I \neq \emptyset$ for every right ideal I of A^*.
- $R = PA^*$ is essential if and only if P is a maximal prefix code.

Thompson-Higman Groups $V_{k,1}$

- $R_f^e(A^*)$:= set of all A^*-isomorphisms between finitely generated essential right ideals of A^*.

- It is an inverse submonoid of $\mathscr{I}_{A^*}$.

- It is an F-inverse monoid, i.e., every σ-class contains a maximum element.

- $V_{k,1}$:= $R_f^e(A^*)/\sigma$.

Note An A^*-isomorphism $\varphi : P_1 A^* \to P_2 A^*$ (P_1, P_2 prefix codes) restricts to a bijection from P_1 to P_2.

Generalisation I

C is a right LCM monoid if C is left cancellative and for $a, b \in C$, $aC \cap bC = \emptyset$ or is principal.

Artin monoids (in particular, free monoids), Garside monoids.

A projective right ideal of C is a disjoint union of principal right ideals. If

$$P = a_1 C \sqcup \cdots \sqcup a_t C$$

is a projective right ideal, say $\{a_1, \ldots, a_t\}$ is a basis for P.
Assumption C has finitely generated essential projective right ideals. Holds if C is a finitely generated monoid.

$R^e_{fp}(C)$:= set of all C-isomorphisms between finitely generated essential projective right ideals of C. It is an inverse submonoid of $\mathscr{I}_C$.

What about $R^e_{fp}(C)/\sigma$?

Generalisation II

C is still a right LCM monoid.

- If C is right Ore and cancellative, then $R^e_{fp}(C)/\sigma$ is the group of right fractions of C.
- If C is a left Rees monoid with finitely generated free part, then $C \cong A^* \bowtie G$ (see next slide) where $A = \{a_1, \ldots, a_k\}$ and G is an appropriate group.
  $R^e_{fp}(C)/\sigma \cong V_k(G)$ (introduced by Nekrashevych).
- If $C = A^* \times \cdots \times A^*$ (n factors), then $R^e_{fp}(C)/\sigma \cong nV_{k,1}$ (introduced by Brin).
- Brown-Stein groups???

Left Rees Monoids

A left cancellative monoid C is a left Rees monoid if all its right ideals are projective, and each principal right ideal is contained in only finitely many principal right ideals.

G, C monoids. Actions: G on C: $(g, c) \mapsto g \cdot c$; C on G: $(g, c) \mapsto g|_c$.
On $C \times G$ define

$$(c, g)(d, h) = (c(g \cdot d), g|_d h).$$

With appropriate conditions on the actions, get a monoid $C \bowtie G$, the Zappa-Szép product of C and G.

Theorem (Lawson).
A monoid M is a left Rees monoid if and only if $M \cong A^* \bowtie G$ for some set A and group G.

In this case, the action of G on A^* is a self-similar action, i.e., $\forall g \in G, a \in A, \exists$ unique $b \in A, h \in G$ such that $g \cdot (aw) = b(h \cdot w)$ for all $w \in A^*$. ($b = g \cdot a$ and $h = g|_a$.)

Alternative view of $R^e_{fp}(C)$: Inverse Hulls

C left cancellative. For $a \in C$, the mapping λ_a defined by

$$\lambda_a(c) = ac.$$

is one-one with domain C. $IH(C) = \mathrm{Inv}\langle \lambda_a : a \in C \rangle$ is the inverse hull of C.

$$IH^0(C) = \begin{cases} IH(C) & \text{if } 0 \in IH(C) \\ IH(C) \cup \{0\} & \text{otherwise.} \end{cases}$$

Theorem (McAlister; also Nivat/Perrot)

The following are equivalent:

1. *$IH^0(C)$ is 0-bsimple;*
2. *every non-zero element of $IH^0(C)$ can be written as $\lambda_a \lambda_b^{-1}$ for some $a, b \in C$;*
3. *the domain of each non-zero element of $IH^0(C)$ is a principal right ideal;*
4. *C is a right LCM monoid.*

Alternative view of $R^e_{fp}(C)$: Orthogonal Completions 1

S inverse semigroup with zero. $a, b \in S$ are orthogonal ($a \perp b$) if

$$a^{-1}b = 0 = ab^{-1}.$$

Clearly, $a \perp b$ iff $aa^{-1} \perp bb^{-1}$ and $a^{-1}a \perp b^{-1}b$.
$A \subseteq S$ is orthogonal if $a \perp b$ for all distinct $a, b \in A$.

S is orthogonally complete if it satisfies:

1. $\{a_1, \ldots, a_n\}$ orthogonal implies $a_1 \vee \cdots \vee a_n$ exists (natural po), and
2. multiplication distributes over joins of finite orthogonal sets.

Examples

1. Symmetric inverse monoids.
2. $IH^0(C)$ where C is a right Ore and right LCM monoid.

Alternative view of $R^e_{fp}(C)$: Orthogonal Completions 2

S inverse semigroup with zero.

$$D(S) = \{A \subseteq S : 0 \in A, |A| < \infty, A \text{ is orthogonal}\}.$$

Theorem (Lawson)

1. *$D(S)$ is an inverse subsemigroup of $P(S)$; it is a monoid if S is a monoid.*
2. *$\iota : S \to D(S)$ given by $a \mapsto \{0, a\}$ embeds S in $D(S)$*
3. *$D(S)$ is orthogonally complete.*
4. *If $\theta : S \to T$ is a homomorphism to an orthogonally complete inverse semigroup T, then there is a unique join preserving homomorphism $\varphi : D(S) \to T$ such that $\varphi\iota = \theta$.*

Say $D(S)$ is the orthogonal completion of S.

Alternative view of $R^e_{fp}(C)$: Orthogonal Completions 3

C is a right LCM monoid. $R_f(C)$ (resp. $R_{fp}(C)$) is the set of C-isomorphisms between finitely generated (resp. finitely generated projective) right ideals of C.
$R_{fp}(C) \subseteq R_f(C)$ are inverse submonoids of the the symmetric inverse monoid on C and $R^e_{fp}(C) \subseteq R_{fp}(C)$.

The polycyclic monoid P_n on $A = \{a_1, \ldots, a_n\}$ is $IH^0(A^*)$ and a presentation for it is:

$$\langle A \cup A^{-1} \mid aa^{-1} = 1; ab^{-1} = 0 \text{ if } a \neq b \rangle.$$

Theorem (Lawson)

$D(P_n) \cong R_f(A^*) = R_{fp}(A^*)$.

Alternative view of $R^e_{fp}(C)$: Orthogonal Completions 4

Recall that

$$IH^0(C) = \{\lambda_c \lambda_d^{-1} : c, d \in C\} \cup \{0\}.$$

Product:

$$(\lambda_a \lambda_b^{-1})(\lambda_c \lambda_d^{-1}) = \begin{cases} \lambda_{as} \lambda_{dt}^{-1} & \text{if } bC \cap cC = mC \text{ with } m = bs = ct \\ 0 & \text{if } bC \cap cC = \emptyset. \end{cases}$$

$\{\lambda_{a_1} \lambda_{b_1}^{-1}, \ldots, \lambda_{a_k} \lambda_{b_k}^{-1}\} \cup \{0\}$ is orthogonal

iff $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ are bases for projective right ideals of C

iff for all i, j with $i \neq j$ we have $a_i C \cap a_j C = \emptyset$ and $b_i C \cap b_j C = \emptyset$.

Alternative view of $R^e_{fp}(C)$: Orthogonal Completions 5

Theorem
$D(IH^0(C)) \cong R_{fp}(C)$.

Idea of proof: Let $A \in D(IH^0(C))$, say
$A = \{\lambda_{a_1}\lambda_{b_1}^{-1}, \ldots, \lambda_{a_k}\lambda_{b_k}^{-1}\} \cup \{0\}$.
Then $I = \{a_1, \ldots, a_k\}C$ and $J = \{b_1, \ldots, b_k\}C$ are projective right ideals and

$$\theta_A : J \to I \text{ given by } (b_i c)\theta_A = a_i c$$

is a C-isomorphism. Now define

$$\theta : D(IH^0(C)) \to R_{fp}(C) \text{ by } \theta(A) = \theta_A$$

and verify that θ is an isomorphism.

Alternative view of $R^e_{fp}(C)$: Orthogonal Completions 6

S is an inverse monoid with zero.

$S^e := \{a \in S : Sa \text{ and } aS \text{ are essential}\}$.
S^e is an inverse submonoid of S called the essential part of S.

An idempotent e is essential if $e \in S^e$. This is true if and only if $ef \neq 0$ for all non-zero idempotents f of S.
$a \in S^e$ if and only if aa^{-1} and $a^{-1}a$ are essential idempotents

The isomorphism θ restricts to an isomorphism

$$D^e(IH^0(C)) \cong R^e_{fp}(C).$$

Right Ore Right LCM Monoids

Let C be right Ore and right LCM. Then every projective right ideal is principal. (Two principal right ideals cannot be disjoint).

So, all orthogonal subsets of $IH^0(C)$ have the form $\{\lambda_a \lambda_b^{-1}, 0\}$; hence the embedding of $IH^0(C)$ into $D(IH^0(C))$ is surjective.

Every nonzero idempotent of $IH^0(C)$ is essential. So

$$D^e(IH^0(C)) = IH(C).$$

Well known that the group of right fractions of C is isomorphic to $IH(C)/\sigma$.

More on Zappa-Szép Products I

Let C be a right LCM monoid with trivial group of units, and G be a group. Suppose we have actions so that we can form $D = C \bowtie G$. Then

1. D is left cancellative;
2. D is right LCM;
3. the group of units of D is $\{(1, g) : g \in G\}$;
4. the partially ordered set of principal right ideals of D is order-isomorphic to the partially ordered set of principal right ideals of C.

More on Zappa-Szép Products II

Remember that for any right LCM monoid B,

$$IH^0(B) = \{\lambda_a \lambda_b^{-1} : a, b \in B\}.$$

$$\lambda_a \lambda_b^{-1} = \lambda_c \lambda_d^{-1} \Leftrightarrow \exists \text{ unit } u \in B \text{ such that } au = c, bu = d.$$

Write elements of $IH^0(B)$ as $\sim$-equivalence classes $[a, b]$ where

$$(a, b) \sim (c, d) \Leftrightarrow \exists \text{ unit } u \in B \text{ such that } au = c, bu = d.$$

Now consider $IH^0(D)$ where $D = C \bowtie G$. Elements are:

$$[(a, g), (b, h)] = [(a, gh^{-1}), (b, 1)]$$

so can represent elements by triples $(a, g, b) \in C \times G \times C$.

More on Zappa-Szép Products III: Orthogonal Subsets

The following are equivalent:

1. $X = \{(a_1, g_1, b_1), \ldots, (a_t, g_t, b_t)\} \cup \{0\}$ is an orthogonal subset of $IH^0(D)$;
2. $\overline{X} = \{\lambda_{a_1}\lambda_{b_1}^{-1}, \ldots, \lambda_{a_t}\lambda_{b_t}^{-1}\} \cup \{0\}$ is an orthogonal subset of $IH^0(C)$;
3. $A = \{a_1, \ldots, a_t\}$ and $B = \{b_1, \ldots, b_t\}$ are bases for projective right ideals of C.

Consequently,

$$\begin{aligned} D^e(IH^0(D)) &= \{X : \overline{X} \in D^e(IH^0(C))\} \\ &= \{X : A, B \text{ are bases for essential projective right ideals}\}. \end{aligned}$$