Hyperbolic and word-hyperbolic semigroups

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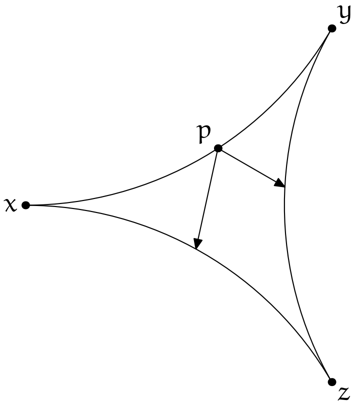
Slim triangles



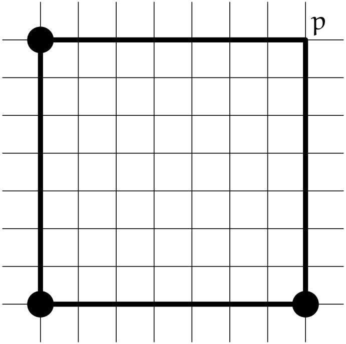
The space is δ-hyperbolic if for any geodesic triangle $\triangle_{xyz}$,

$$p \in [xy] \implies d(p, [yz] \cup [zx]) \leqslant \delta.$$

A group G generated by X is hyperbolic if the Cayley graph $\Gamma(G, X)$ is a hyperbolic metric space.

- Trees are 0-hyperbolic, so free groups are hyperbolic.

$\mathbb{Z}^2$ is not hyperbolic



The point p can be very far from the other sides of geodesic triangles like this.

Quasi-isometries

Let (S, d_S), (T, d_T) be metric spaces.

A map $\phi : S \to T$ is a quasi-isometry if there are $m, c, k \geqslant 0$ such that

$$\frac{1}{m} d_S(x, y) - c \leqslant d_T(x\phi, y\phi) \leqslant m\ d_S(x, y) + c;$$

and such that every point in T is at most k from some point in $S\phi$.

Hyperbolicity is preserved under quasi-isometries.

If G is generated by both X and Y then $\Gamma(G, X)$ and $\Gamma(G, Y)$ are quasi-isometric. Hence hyperbolicity is independent of the choice of generating set.

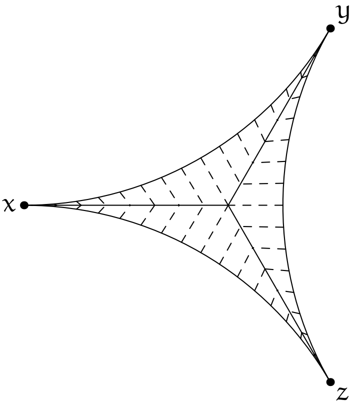
Other equivalent definitions of hyperbolic groups

- Admitting a finite Dehn's presentation.
- Having linear isoperimetric inequality.
- Gilman's linguistic characterization: G is hyperbolic if there is a regular language $L \subseteq X^*$ such that L maps onto G and such that

$$M(L) = \{u\#_1 v\#_2 w^{\mathrm{rev}} : u, v, w \in L, uv =_G w\}$$

is context-free.

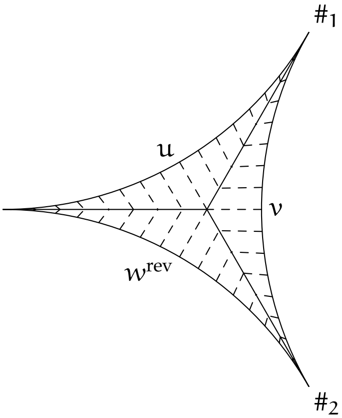
Thin triangles



For any geodesic triangle $\triangle_{xyz}$, there is a unique map $f : \triangle_{xyz} \to T_{xyz}$, where T_{xyz} is a tripod connecting x, y, z, such that f restricts to an isometry on edge of $\triangle_{xyz}$.

The space is δ-hyperbolic if for any geodesic triangle $\triangle_{xyz}$, the preimage of any point of T_{xyz} has diameter at most δ.

Context-free grammar describing thin triangles



Language of geodesics is regular (Cannon).

Non-terminals of CFG record word differences of elements mapping to the same element of the tripod.

A semigroup S generated by X is hyperbolic if $\Gamma(S, X)$ is hyperbolic.

A semigroup S generated by X is word-hyperbolic if there is a regular language $L \subseteq X^+$ such that

$$M(L) = \{u\#_1 v\#_2 w^{\text{rev}} : u, v, w \in L, uv =_S w\}$$

is context-free. The pair $(L, M(L))$ is a word-hyperbolic structure for S.

These are *not* equivalent for semigroups.

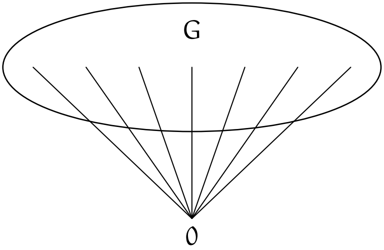
Hyperbolic vs. word-hyperbolicity

Proposition (Duncan & Gilman 2004)

S is word-hyperbolic $\iff$ S^0 is word-hyperbolic.

Let G be a non-hyperbolic group.

- G^0 is not word-hyperbolic.
- G^0 is hyperbolic:



Finite geometric type

A f.g. semigroup has finite geometric type if there is a bound on the number of in-edges at any vertex of its Cayley graph.

Theorem (C.)

Let S be a monoid of finite geometric type and let T be a finite Rees index submonoid of S. Then the natural embedding map $T \hookrightarrow S$ is a quasi-isometry.

Hyperbolic vs. word-hyperbolicity

Does hyperbolicity + some extra geometric condition imply word-hyperbolicity?

Example (C., Gray, Malheiro)

There exists a monoid that has the following properties:

- Quasi-isometric to a tree (and so hyperbolic).
- Right-cancellative.
- Insoluble word problem (and so not word-hyperbolic).

Question

Does hyperbolicity + left-cancellativity or cancellativity imply word-hyperbolicity?

Special rewriting systems

Theorem (Cassaigne & Silva 2009)

Any monoid presented by a confluent finite special rewriting system is hyperbolic and word-hyperbolic.

(Special rewriting system: RHS of any rule is ε.)

Monadic rewriting systems

Theorem (C. 2010)

Any monoid presented by a confluent regular monadic rewriting system is hyperbolic and word-hyperbolic.

(Monadic rewriting system: RHS of any rule is ε or a single letter.)

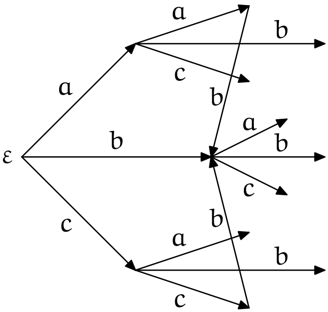
Idea of proof

Let $(A, \mathcal{R})$ be a confluent monadic rewriting system presenting M.
Identify M with the language of normal form words.
2 types of edge in $\Gamma(M, A)$:

1. $u \xrightarrow{a} ua$;
2. $u \xrightarrow{a} v$, where $ua \Rightarrow^{+} v$.

Let Σ be the subgraph with only type 1 edges.
Σ is a subgraph of $\Gamma(A^*, A)$ and thus a tree.

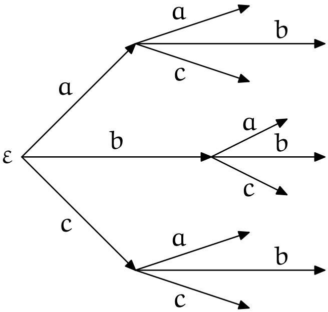
Idea of proof



Graph Γ

$$\begin{aligned} A &= \{a, b, c\}, \\ \mathcal{R} &= \{a^2b \to b, c^2b \to b\}. \end{aligned}$$

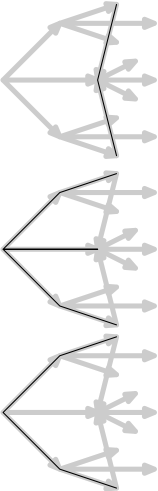
Idea of proof



Graph Σ

$$\mathcal{A} = \{a, b, c\},$$
$$\mathcal{R} = \{a^2b \to b, c^2b \to b\}.$$

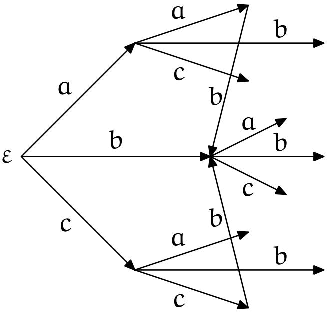
Idea of proof



Γ-geodesic α

Path α' in Σ

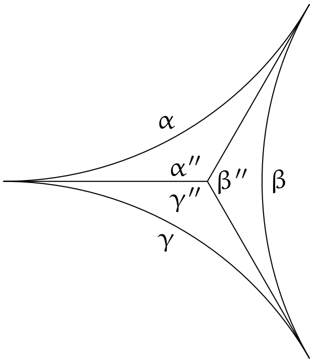
Σ-geodesic α''



Graph Γ

But every vertex on α' (and thus α'') is at most $n+1$ from some vertex on α.
And every vertex on α is at most $3n+1$ from some vertex on α''.

Idea of proof



An even simpler proof in the finite special case

Suppose $u \xrightarrow{a} v$ is an edge not in Σ. Then $ua \Rightarrow^* v$. So v is a prefix of u and $||u| - |v|| \leqslant n$. So $d_\Sigma(u, v) \leqslant n$.

So if there is a path of length k joining u and v in $\Gamma(S, A)$, there is a path of length kn joining them in Σ.

So $d_\Gamma(u, v) \leqslant d_\Sigma(u, v) \leqslant n d_\Gamma(u, v)$.

So the embedding map $\Sigma \hookrightarrow \Gamma(S, A)$ is a quasi-isometry.

Context-free special rewriting systems

Proposition (C. & Maltcev 2011)

Any monoid presented by a confluent context-free monadic rewriting system $(A, \mathcal{R})$ *admits a word-hyperbolic structure* $(A^*, M(A^*))$.

Context-free special rewriting systems

- The language of words of the form

$$\$_{a_1} \cdots \$_{a_k} \#_1 \$_{a_{k+1}} \cdots \$_{a_n} \#_2 \$_{a_n} \cdots \$_{a_1}$$

  is defined by a context-free grammar Γ.
- Extend Γ to allow derivations $\$_{a_i} \Rightarrow_{\Gamma}^{*} w$, whenever $w \Rightarrow_{\mathcal{R}}^{*} a_i$.
- This grammar defines $M(A^*)$.

Word-hyperbolicity with uniqueness

Every hyperbolic group G admits a word-hyperbolic structure with uniqueness $(L, M(L))$ where L maps bijectively onto G. Duncan & Gilman (2004) asked whether a word-hyperbolic semigroup always admits a word-hyperbolic structure with uniqueness.

Example (C. & Maltcev 2011)

The monoid presented by $\langle A \mid \mathcal{R} \rangle$, where $A = \{a, b, c, d\}$ and

$$\mathcal{R} = \{(ab^\alpha c^\alpha d, \varepsilon) : \alpha \in \mathbb{N} \cup \{0\}\}$$

is word-hyperbolic but does not admit a regular language of unique normal forms.

Word-hyperbolicity with uniqueness

Question

If a semigroup admits a word-hyperbolic structure $(L, M(L))$ where $M(L)$ is a deterministic context-free language, must it admit a word-hyperbolic structure with uniqueness?

(Word-hyperbolic groups always admit a word-hyperbolic structure where $M(L)$ is deterministic.)

Word problem

Let $(L.M(L)$ be a word-hyperbolic structure for a monoid M. Let $e \in L$ represent 1_M.

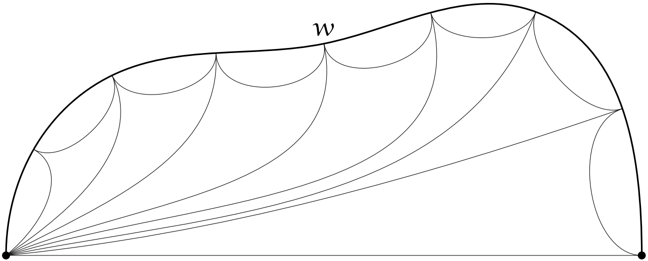
Given two words $w, w' \in A^*$, compute $u, u' \in L$ with $w =_M u$ and $w' =_M u'$, then check whether $u\#_1 e\#_2 (u')^{\text{rev}} \in M(L)$.

Checking membership of a CFL takes cubic time.

Computing representatives in L

Lemma (Hoffmann, Kuske, Otto, Thomas)

Given non-empty $p, q \in L$*, one can compute* $r \in L$ *satisfying* $pq =_M r$ *with* $|r| \leqslant c(|p| + |q|)$ *in time* $\mathcal{O}((|p| + |q|)^5)$*.*

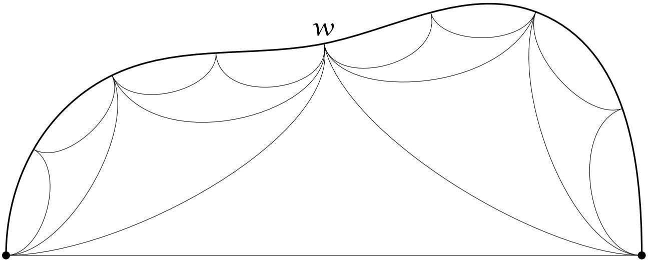


For each $a \in A$, there is $s_a \in L$ with $s_a =_M a$.
Let $w = w_1 \cdots w_n$. Compute $u_{i+1} = u_i s_{w_{i+1}}$.
Then $|u_i| \leqslant d^i$ for some d.
So this takes exponential time.

A better algorithm

Lemma (Hoffmann, Kuske, Otto, Thomas)

Given non-empty $p, q \in L$, *one can compute* $r \in L$ *satisfying* $pq =_M r$ *with* $|r| \leqslant c(|p| + |q|)$ *in time* $\mathcal{O}((|p| + |q|)^5)$.



Let $w = w_1 \cdots w_n$. Multiply adjacent elements.
There are $\log n$ iterations. Length increase of c each iteration.
So overall length increase is $c^{\log n} = n^{\log c}$.
So this takes polynomial time.