Some results on almost factorizable semigroups

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Inverse semigroups

S — inverse semigroup
E — semilattice of idempotents of S
σ — least group congruence on S

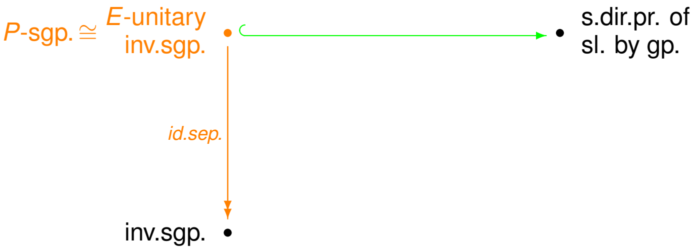
Definition

S is E-unitary

$$\overset{\text{def}}{\iff} \quad e \leq a \text{ implies } a \in E \text{ for every } e \in E, a \in S,$$
$$\overset{\text{def}}{\iff} \quad \operatorname{Ker} \sigma = E,$$
$$\overset{\text{def}}{\iff} \quad a \, (\mathcal{R} \cap \sigma) \, b \text{ implies } a = b \text{ for every } a, b \in S$$

Inverse semigroups

McAlister ('74)
O'Carroll ('76)



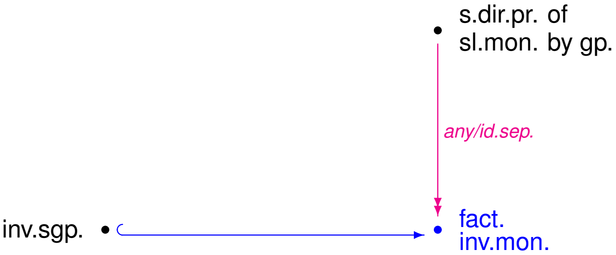
M — inverse monoid with identity 1
E — semilattice of idempotents of M
U — group of units of M (i.e. the $\mathcal{H}$-class of 1)

Definition

M is factorizable
$\overset{\text{def}}{\Longleftrightarrow}$ $M = EU$
$\overset{\text{def}}{\Longleftrightarrow}$ for every $a \in M$, there exists $u \in U$ with $a \leq u$

Inverse semigroups

Chen, Hsieh ('74)
McAlister, Reilly ('77)



Inverse semigroups

An analogue for inverse *semigroups*?

S — inverse semigroup
E — semilattice of idempotents of S
$\mathrm{P}(S)$ — monoid of partial 1-1 right translations of S

H — non-empty subset of S

Definition

H is a permissible set

$\overset{\text{def}}{\iff}$ H is an order ideal with respect to $\leq$, and
$a^{-1}b, ab^{-1} \in E$ for every $a, b \in H$

Inverse semigroups

$C(S)$ — set of all permissible subsets of S

Fact

$C(S)$ forms an inverse monoid with respect to usual set product, and it is isomorphic to $\mathrm{P}(S)$.

$U\mathrm{P}(S)$ — group of units of $\mathrm{P}S$
$UC(S)$ — group of units of $C(S)$

Lawson ('94)

Definition

S is almost factorizable

$\overset{\text{def}}{\iff}$ for every $a \in S$, there exists $\rho \in U\mathcal{P}(S)$ with $a \in E\rho$

$\overset{\text{def}}{\iff}$ for every $a \in S$, there exists $H \in UC(S)$ with $a \in H$

Results

Let M be an inverse monoid.

1. *M is almost factorizable iff it is factorizable.*
2. *If M is factorizable then $M \setminus U$ is an almost factorizable inverse semigroup, and each almost factorizable inverse semigroup is of this form.*

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Lawson ('94) (c.f. also McAlister ('76))

Summary of the inverse case

Result

An inverse semigroup is E-unitary and almost factorizable iff it is isomorphic to a semidirect product of a semilattice by a group.

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Orthodox semigroups

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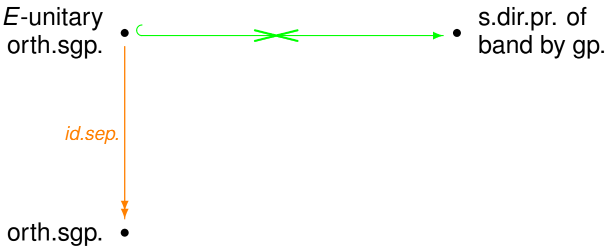
Orthodox semigroups

Takizawa ('79), Sz. ('80)
Billhardt ('98)



Orthodox semigroups

Sz. ('93)

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Fact

M is factorizable iff it is an (id.sep.) homomorphic image of a semidirect product of a band monoid by a group.

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S — orthodox semigroup
E — band of idempotents of S
$U\Omega(S)$ — group of units of translational hull of S

Fact

If S is inverse then $U\Omega(S)$ is isomorphic to $U\mathrm{P}(S)$.

Hartmann ('07, PhD Thesis)

Definition

S is almost factorizable
$\overset{\text{def}}{\iff}$ for every $a \in S$, there exists $(\lambda, \rho) \in U\Omega(S)$ with $a \in E\rho$
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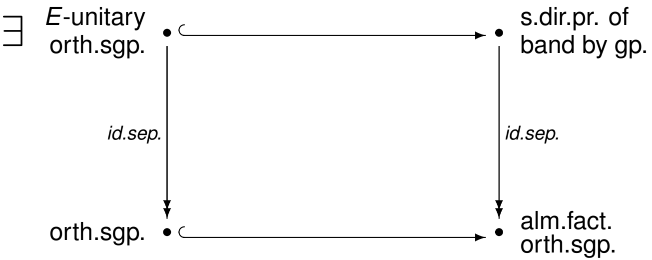

Fact

An orthodox semigroup isomorphic to a semidirect product of a band by a group is E-unitary and almost factorizable.

Question. Does the converse hold?

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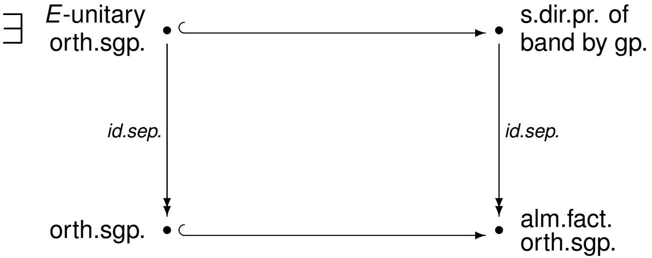

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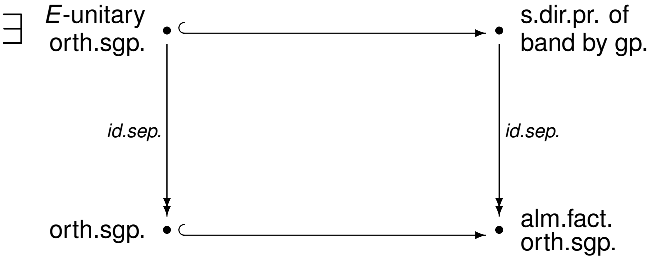

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Question. Does the converse hold?

Hartmann, Sz. (subm.)

Answer. No.

Example

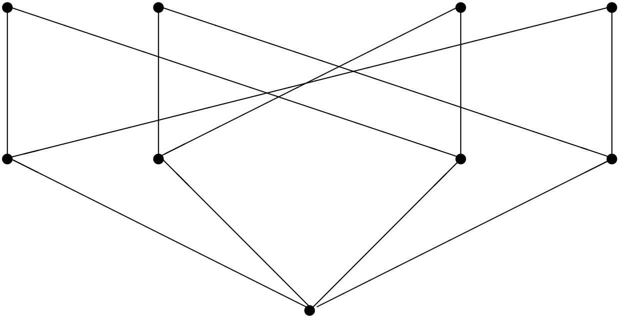
$S = B \rtimes \mathbb{Z}_4$ — semidirect product with B a left normal band

κ — idempotent pure congruence on S s.t.

- the greatest group homomorphic image of S/κ is $\mathbb{Z}_2$
- S/κ is not isomorphic to a semidirect product of a band by a group

Orthodox semigroups

structure semilattice of B:



Orthodox semigroups

S — orthodox semigroups
γ — least inverse semigroup congruence on S

$\chi\colon U\Omega(S) \to U\Omega(S/\gamma),\ (\lambda,\rho)\chi = (\lambda_\gamma,\rho_\gamma)$
where e.g. $\lambda_\gamma(s\gamma) = (\lambda s)\gamma$ $(s \in S)$
is a group homomorphism

S is E-unitary and almost factorizable $\Longrightarrow$ χ is surjective
$\Longrightarrow$ $U\Omega(S)$ is an extension of $\operatorname{Ker}\chi$ by $U\Omega(S/\gamma)$

Theorem

S is isomorphic to a semidirect product of a band by a group iff S is E-unitary, almost factorizable, and the group extension determined by χ is splitting.

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Left restriction semigroups

$\mathcal{PT}_X$ — monoid of all partial transformations on X
$\mathcal{I}_X$ — monoid of all partial 1-1 transformations on X
$^+$ — unary operation: $\alpha^+ \stackrel{\text{def}}{=} \mathrm{id}_{\mathrm{dom}\,\alpha}$ (d. idempotents)
$\leq$ — natural partial order

Definition

$S = (S; \cdot, ^+)$ is a left restriction semigroup
$\stackrel{\text{def}}{\Longleftrightarrow}$ S is isomorphic to a $(2, 1)$-subalgebra of $\mathcal{PT}_X = (\mathcal{PT}_X; \cdot, ^+)$

$S = (S; \cdot, ^+)$ is a left ample
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Left restriction semigroups

S — left restriction semigroup
$E \stackrel{\text{def}}{=} \{a^+ : a \in S\}$ — semilattice of d. idempotents of S
σ — least (monoid) congruence on S where E is within a class

in particular:

S — left ample semigroup
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σ — least right cancellative (monoid) congruence on S

Definition

S is proper

$\stackrel{\text{def}}{\Longleftrightarrow}$ $a^+ = b^+$ and $a \,\sigma\, b$ imply $a = b$ for every $a, b \in S$

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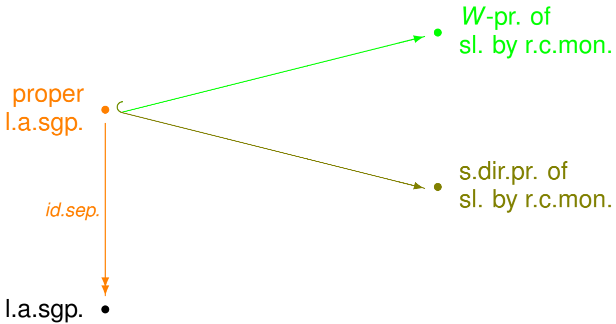
Left restriction semigroups

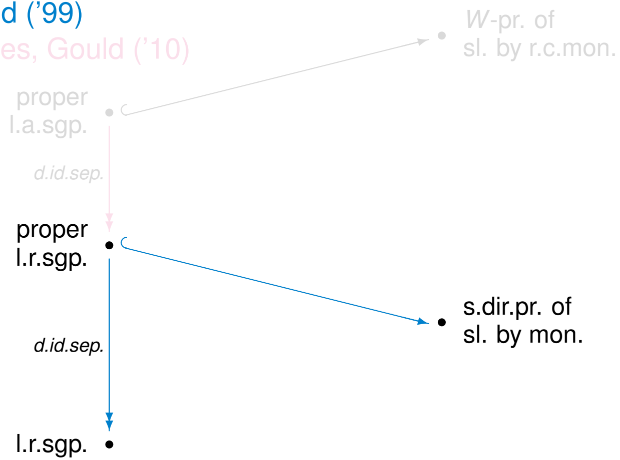
Fountain ('77)
Fountain, Gomes ('93)
Billhardt ('97)



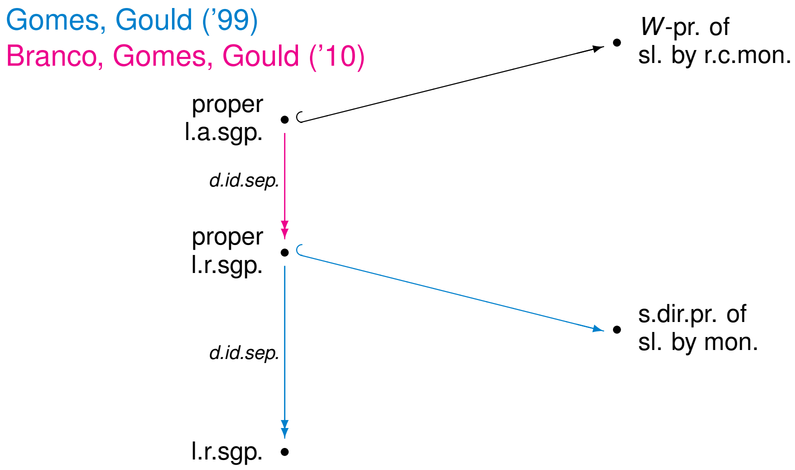
Left restriction semigroups

Gomes, Gould ('99)

Branco, Gomes, Gould ('10)



Left restriction semigroups



Left restriction semigroups

Y — semilattice
M — monoid with identity 1
M acts on Y on the right s.t. for any $a \in M,\ x, y \in Y$

$$x^a = y^a \implies x = y$$
$$x \leq y^a \implies (\exists z \in Y)\ x = z^a$$

Definition

$W(M, Y) \stackrel{\text{def}}{=} \{(a, y^a) : a \in M, y \in Y\} \leq M \ltimes Y$ with

$$(a, y^a)^+ \stackrel{\text{def}}{=} (1, y)$$

Facts

1. *$W(M, Y)$ is a proper left restriction semigroup.*
2. *$W(M, Y)$ is a proper left ample semigroup iff M is right cancellative.*

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Left restriction semigroups

M — left ample monoid with identity 1
E — semilattice of idempotents of M
$R \stackrel{\text{def}}{=} \{r \in M : r^+ = 1\}$, a right cancellative submonoid in M

El Qallali ('81)

Definition

M is factorizable
$\stackrel{\text{def}}{\Longleftrightarrow}$ $M = ER$
$\stackrel{\text{def}}{\Longleftrightarrow}$ for every $a \in M$, there exists $r \in R$ with $a \leq r$

El Qallali, Fountain ('05)

l.a.sgp. $\bullet \hookrightarrow \bullet$ fact. l.a.mon.

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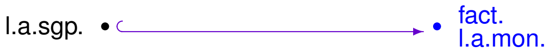

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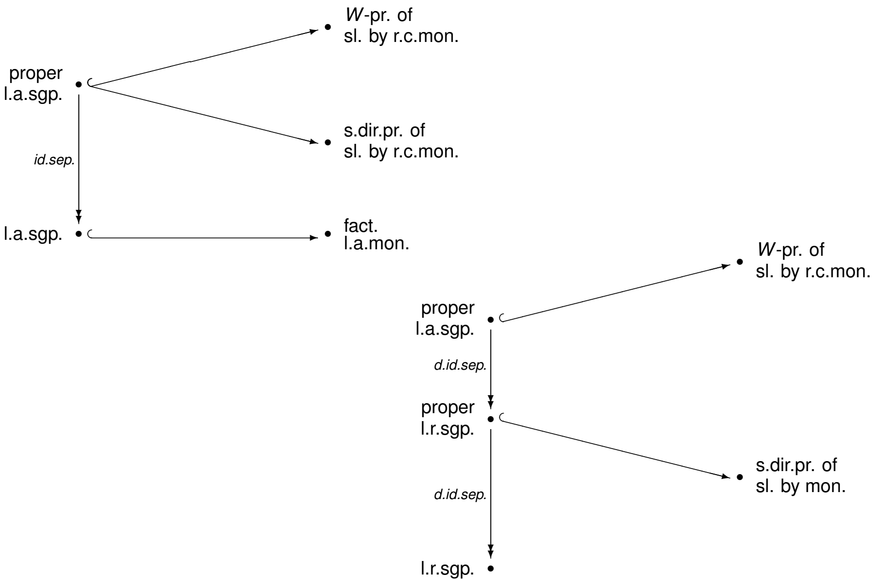
Summary of the left ample/restriction case



Restriction semigroups

Dual of a left ample/restriction semigroup:

$S = (S; \cdot, ^*)$ — right ample/restriction semigroup

Definition

1. $S = (S; \cdot, ^+, ^*)$ is an ample/restriction semigroup
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   $(S; \cdot, ^*)$ is right ample/restriction, and
   $E = \{a^+ : a \in S\} = \{a^* : a \in S\}$
2. $S = (S; \cdot, ^+, ^*)$ is proper
   $\stackrel{\text{def}}{\Longleftrightarrow}$ both $(S; \cdot, ^+)$ and $(S; \cdot, ^*)$ are proper

Fact

$W(M, Y) \leq M \ltimes Y$, and so $(a, y^a)^* \stackrel{\text{def}}{=} (1, y^a)$ makes $W(M, Y)$ a proper restriction semigroup.

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Gomes, Sz. ('07)

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$C(S)$ — restriction monoid of all permissible subsets of S, with identity E

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$C(S)$ — restriction monoid of all permissible subsets of S, with identity E

$RC(S) \stackrel{\text{def}}{=} \{H \in C(S) : H^+ = E\}$, a submonoid in $C(S)$

Definition

S is almost left factorizable
$\stackrel{\text{def}}{\Longleftrightarrow}$ for every $a \in S$, there exists $H \in RC(S)$ with $a \in H$

Results

1. *S is almost left factorizable iff it is any/d. id. sep. homomorphic image of a W-product of a semilattice by a monoid.*
2. *S is isomorphic to a W-product of a semilattice by a monoid iff it proper and almost left factorizable.*

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Fountain, Gomes, Gould ('09)

Left restriction semigroups revisited