Varieties of P-restriction semigroups.

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Background on left restriction semigroups, *aka* **weakly left E-ample semigroups.**

One view: consider the semigroup of partial transformations $\mathcal{PT}_X$ on a set as a unary semigroup under the additional unary operation $^+$, where $\alpha^+ = 1_{\mathsf{dom}\alpha}$. The left restriction semigroups are the abstractions of the (unary) semigroups of partial transformations. Notice that the set E of partial identity maps is a semilattice that is a proper subset of the set of idempotents of $\mathcal{PT}_X$.

An alternative view is that S is a semigroup with a designated subsemilattice E of idempotents, S is weakly left E-adequate, $\widetilde{\mathcal{R}}_E$ is a left congruence and the left ample condition $ae = (ae)^+a$ is satisfied for all $e \in E$.

From yet another point of view — and the one of this talk — the left restriction semigroups are the unary semigroups $(S, \cdot, ^+)$ that are induced from inverse semigroups $(S, \cdot, ^{-1})$ by setting

$$a^+ = aa^{-1}$$

From whichever origin, as unary semigroups they are defined by the identities [Cockett and Lack, 2002; Gould "notes" 2009]:

$$x^+x = x, \quad x^+x^+ = x^+, \quad (xy)^+ = (xy^+)^+,$$

$$x^+y^+ = y^+x^+, \quad xy^+ = (xy)^+x.$$

The right restriction semigroups are defined dually. An inverse semigroup induces a right restriction semigroup by setting $a^* = a^{-1}a$.

A restriction semigroup is both a left and right restriction semigroup, with respect to a common set E.

We regard it as a 'bi-unary' semigroup $(S, \cdot, ^+, ^*)$, the operations being attached to a common subsemilattice E.

So every inverse semigroup induces a restriction semigroup by setting $a^+ = aa^{-1}$ and $a^* = a^{-1}a$.

At the opposite extreme, every monoid $(S, \cdot, 1)$ induces a 'reduced' restriction semigroup by setting

$$a^+ = 1 = a^*.$$

Generalizing restriction semigroups.

First of all, we want to retain ‘adequacy’. In the past, this was approached by allowing E to be a band instead of a semilattice.

Rather than using E itself as the focus, we consider semigroups obtained by inducing one or both of the operations $a^+ = aa^{-1}$ and $a^* = a^{-1}a$ from a ‘nice’ class of semigroups endowed with an inversion operation.

Now E is just the set of ‘projections’, so we prefer to denote it P_S.

A *regular* $*$*-semigroup* [Nordahl and Scheiblich, 1978] is a semigroup $(S, \cdot, ^{-1})$ with a regular involution:

$$xx^{-1}x = x, \quad x^{-1}xx^{-1} = x^{-1}$$

$$(x^{-1})^{-1} = x, \quad (xy)^{-1} = y^{-1}x^{-1}.$$

Under the signature $(\cdot, ^{-1})$, regular $*$-semigroups form a variety, denoted $\mathbf{RS}$. Well-known subvarieties include groups, $\mathbf{G}$, inverse semigroups, $\mathbf{I}$, and orthodox $*$-semigroups, $\mathbf{O}$.

On any regular $*$-semigroup, unary operations $a^+ = aa^{-1}$, $a^* = a^{-1}a$ are induced, as above. Now $P_S = \{a^+ : a \in S\} = \{a^* : a \in S\}$ is the usual set of projections, in the standard terminology.

The induced unary semigroup $(S, \cdot, ^+)$ satisfies:

$x^+ x = x, \quad x^+ x^+ = x^+, \quad (xy)^+ = (xy^+)^+,$
$(x^+ y)^+ = x^+ y^+ x^+.$

The last identity is purely a consequence of the involutory property.

The induced unary semigroup $(S, \cdot, ^*)$ satisfies the dual identities and shares the same set of projections.

The bi-unary semigroup $(S, \cdot, ^+, ^*)$ further satisfies the 'generalized left and right ample' identities

$$(xy)^+ x = xy^+ x^*, \quad x(yx)^* = x^+ y^* x.$$

Again, these are consequences of the involutory property only.

A P-*restriction semigroup* is a bi-unary semigroup $(S, \cdot, ^+, ^*)$ that satisfies the identities in the previous slide. Then (it turns out that) the restriction semigroups are the P-restriction semigroups for which the set P_S of projections forms a semilattice. In general, P_S is not a subsemigroup of S, but can be characterized abstractly as a 'projection algebra'.

With every projection algebra P is associated a 'generalized Munn semigroup' T_P, which is a fundamental regular $*$-semigroup.

Theorem For any P-restriction semigroup S, there is a P-separating $(^+,^*)$-representation θ of S onto a full subsemigroup of the regular $*$-semigroup T_{P_S}.

Theorem For any P-restriction semigroup S, the subsemigroup $\langle P_S \rangle$ generated by the projections is a regular $*$-semigroup, which we call the P-core, C_S, of S. If S is induced from a regular $*$-semigroup, this is the usual (idempotent-generated) core.

We can consider P-restriction semigroups under the signature $(\cdot, ^+, ^*)$. Let $\mathbf{PR}$ denote the variety of P-restriction semigroups.

Since every regular $*$-semigroup $(S, \cdot, ^{-1})$ induces the P-restriction semigroup $(S, \cdot ^+, ^*)$, every variety $\mathbf{V}$ of regular $*$-semigroups induces a variety $\mathcal{P}(\mathbf{V})$ of P-restriction semigroups.

$\mathcal{P}(\mathbf{V})$ comprises those that $(^+, ^*)$-*divide* some member of $\mathbf{V}$.

Question: is $\mathcal{P}(\mathbf{RS}) = \mathbf{PR}$?

That is, do the identities on the previous slide characterize the bi-unary semigroups induced from regular $*$-semigroups?

More generally, given $\mathbf{V}$, what is $\mathcal{P}(\mathbf{V})$?

It is known (implicitly, at least) that the variety $\mathbf{I}$ of inverse semigroups induces the variety $\mathbf{R}$ of restriction semigroups; the variety $\mathbf{G}$ of groups induces the variety of reduced restriction semigroups ($x^{+} = x^{*} = 1$).

Note that $\mathbf{I}$ and $\mathbf{R}$ comprise respectively the regular $*$-semigroups and the P-restriction semigroups whose P-core is a semilattice.

We can recognize, or define, many interesting varieties in this way.

For *any* variety $\mathbf{V}$ of regular $*$-semigroups:

- let $C\mathbf{V}$ comprise the *regular $*$-semigroups* whose cores belong to $\mathbf{V}$;

- let $\mathbf{P}C\mathbf{V}$ comprise the *P-restriction* semigroups whose cores belong to $\mathbf{V}$.

If $\mathbf{V} = \mathbf{T}$ (trivial semigroups), then $C\mathbf{T}$ comprises the groups and $\mathbf{P}C\mathbf{T}$ comprises the reduced restriction semigroups.

If $\mathbf{V} = \mathbf{SL}$ (semilattices), then $C\mathbf{SL}$ comprises inverse semigroups and $\mathbf{P}C\mathbf{SL}$ comprises the restriction semigroups.

If $\mathbf{V} = \mathbf{B}$ ($*$-bands), then $C\mathbf{B}$ comprises orthodox $*$-semigroups and $\mathbf{P}C\mathbf{B}$ defines the orthodox P-restriction semigroups.

And if $\mathbf{V} = \mathbf{RS}$, then $C\mathbf{V} = \mathbf{RS}$ and $\mathbf{P}C\mathbf{V} = \mathbf{PR}$.

The original question 'is $\mathcal{P}(\mathbf{RS}) = \mathbf{PR}$?' and all the examples given above fall within the scope of:

Question: When does the equality $\mathcal{P}(C\mathbf{V}) = \mathbf{P}C\mathbf{V}$ hold?

Equivalently: when does every P-restriction semigroup whose P-core belongs to $\mathbf{V}$ divide a regular $*$-semigroup with the same property?

Theorem (Dirty trick) Any P-*fundamental* member of $\mathbf{P}C\mathbf{V}$ actually *embeds* in a member of $C\mathbf{V}$.

Proof. For such a semigroup S, the 'Munn' representation $\theta : S \longrightarrow T_{P_S}$ is faithful.

Further, it maps the P-core of S upon the core of the regular $*$-semigroup T_{P_S}. Hence the latter also belongs to $C\mathbf{V}$. $\square$

Corollary. If the (relatively) free P-restriction semigroup $F\mathbf{P}C\mathbf{V}_X$ is P-fundamental, then

$$\mathcal{P}(C\mathbf{V}) = \mathbf{P}C\mathbf{V}.$$

Application. If $\mathbf{W}$ is any variety of $*$-bands, then

$$\mathcal{P}(C\mathbf{W}) = \mathbf{P}C\mathbf{W}.$$

That is, any (orthodox) P-restriction semigroup whose projections generate a member of $\mathbf{W}$ divides a regular (orthodox) $*$-semigroup with that property.

Without dirty tricks.

Using Rees matrix representations: every P-restriction semigroup whose core is completely simple divides a completely simple $*$-semigroup, so the equality holds for $\mathbf{V} = \mathbf{CS}$.

In general, the equality $\mathcal{P}(C\mathbf{V}) = \mathbf{P}C\mathbf{V}$ holds if and only if

$$F\mathbf{P}C\mathbf{V}_X \cong F\mathcal{P}(C\mathbf{V})_X.$$

Theorem. (By universal algebraic abstract nonsense.) For any variety $\mathbf{V}$ of regular $*$-semigroups, the free P-restriction semigroup $F\mathcal{P}(\mathbf{V})_X$ in the variety induced by $\mathbf{V}$ embeds in the free regular $*$-semigroup $F\mathbf{V}_X$.

In fact, it is isomorphic to the $(^+,^*)$-subsemigroup generated by X. Moreover, this is the subsemigroup generated by X together with the projections of $F\mathbf{V}_X$.

As a result, if (and only if) $\mathcal{P}(C\mathbf{V}) = \mathbf{P}C\mathbf{V}$ holds, $F\mathbf{P}C\mathbf{V}_X$ can be explicitly identified within the associated free regular $*$-semigroup. For example, in the case of $*$-varieties of bands, the structure of the latter is known (Scheiblich, Kad̄ourek and Szendrei).

In general, because the 'Munn' semigroup associated with $F\mathbf{P}C\mathbf{V}_X$ belongs to $C\mathbf{V}$, the map

$$F\mathbf{P}C\mathbf{V}_X \longrightarrow F\mathcal{P}(C\mathbf{V})_X$$

is always P-separating. It follows that the projection algebras of $F\mathbf{P}C\mathbf{V}_X$ and $FC\mathbf{V}_X$ are isomorphic.

Questions:

Does the positive answer for orthodox and for completely simple $*$-semigroups extend to the E-solid case?

Does every P-restriction semigroup divide a regular $*$-semigroup?

Who knows?

Can we go beyond regular $*$-semigroups? E.g. varieties of involutory semigroups, or of regular unary semigroups?

Can we go from 'P-adequacy' to 'P-abundancy', via 'existence varieties'?