Generalizing the Rees Theorem

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The Rees Theorem

We begin by describing an important construction.

Let S be a semigroup, I, Λ be nonempty sets, and P be a $\Lambda \times I$ matrix with entries $p_{\lambda i}$ from S. Define

$$M = M(S; I, \Lambda; P)$$

with the multiplication

$$(i, s, \lambda)(j, t, \mu) = (i, sp_{\lambda j}t, \mu).$$

Then M is a semigroup called a *Rees matrix semigroup over S.*

This is an important technique in semigroup theory for constructing new semigroups from old ones.

The Rees Theorem

Let S be a semigroup. Then S is *simple* if it does not have any proper ideals. Let e be an idempotent in a semigroup S. Then e is a *primitive* Idempotent if for each idempotent f in S such that $f \leq e$ then $f = e$.
A semigroup S is *completely simple* if it is simple and it has a primitive idempotent.

- Theorem (David Rees 1940)

A semigroup is completely simple if and only if it is isomorphic to a Rees matrix semigroup over a group.

Morita Theory

A semigroup S is said to have *local units* if for each element s in S there exist idempotents e and f such that $s = es = sf$.

- Examples

All monoids and all regular semigroups have local units.

We shall not give the full definition of the Morita equivalence here because it is complicated, but the following proposition due to Sunil Talwar can be taken as a definition for the work discussed in this talk.

Morita Theory

- Proposition

Let S be a semigroup with local units. Then S is **Morita equivalent** *to the monoid T iff there is an idempotent e such that S = SeS and eSe is isomorphic to T.*

Talwar proved the following result in 1992.

- Theorem

A semigroup is Morita equivalent to a group iff it is completely simple.

Our goal is to generalize the Rees theorem within the framework of Morita theory.

Unipotent monoids

A monoid is said to be *unipotent* if it contains exactly one idempotent. All groups are unipotent and all cancellative monoids.

A Rees matrix semigroup $M = M(S; I, \Lambda; P)$ over a monoid S is said to be *classical* if each row and each column of P contains an invertible element.

Let S and T *be semigroups with local units*. Then a homomorphism θ from S to T is said to be a *local isomorphism* if $\theta|eSf : eSf \to \theta(e)T\theta(f)$ is an isomorphism for all idempotents e and f *from* S.

Unipotent monoids

- **Covering Theorem**

Let S be semigroup with local units. Then S is Morita eqivalent to a unipotent monoid T iff there is a classical Rees matrix semigroup

$$M = M(T; I, \Lambda; P)$$

with regularity separating surjective local isomorphism $\theta : M \to S$.

A *regularity separating* homomorphism is a homomorphism inducing a bijection between the regular elements

Unipotent monoids

Let S be a semigroup with local units. Then S is a semigroup with *strong local units* if

(i) whenever e and f are idempotents in S such that $e\,x = x$ and $f\,x = x$ then $e\,R\,f$.

(ii) whenever e and f are idempotents in S such that $x\,e = x$ and $x\,f = x$ then $e\,L\,f$.

Special case

If we add an extra condition on S which we have just defined we will have an isomorphism instead of a covering local isomorphism.

- **Theorem**

A semigroup S with local units is isomorphic to a classical Rees matrix semigroup over a unipotent monoid iff it is Morita equivalent to a unipotent monoid and it has strong local units.

Unambiguous semigroups

A semigroup S is said to be *unambiguous* if whenever $a, b, c \in S$ are such that $S^1a \subseteq S^1b$ and $S^1a \subseteq S^1c$ then either $S^1b \subseteq S^1c$ or $S^1c \subseteq S^1b$ and the same for the right ideals. This definition is due to Birget.

Proposition *A classical Rees matrix semigroup* $M = M(S; I, \Lambda; P)$ *over a monoid S is unambiguous if and only if the monoid S is unambiguous.*

From the above proposition, we deduce that completely simple semigroups are unambiguous.

Unambiguous semigroups

Let S be a semigroup with local units. Then S has the property P *locally* iff each local submonoid eSe of S has this property.

Lemma

Let S be a semigroup with local units. Suppose that S is locally unipotent. If S is unambiguous then S has strong local units.

If we combine our results above we have proved the following generalization of the Rees theorem.

Theorem

Let S be an unambiguous semigroup with local units. Then S is Morita equivalent to an unambiguous unipotent monoid iff S is isomorphic to a classical Rees matrix semigroup over an unambiguous unipotent monoid.

CONCLUSION

As we have seen, a completely simple semigroup is unambiguous. It is also locally inverse. One of our goals is to study the structure of unambiguous locally inverse regular semigroups.