Adequate Transversals of Abundant Semigroups

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Outline

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Largest Inverses

- Let S be a regular semigroup with set of idempotents E and let $\leq$ be a partial order on S. Then $(S, \leq)$ is said to be *naturally ordered* if

$$e = ef = fe \text{ implies } e \leq f$$

- If S has a greatest idempotent then for all $x \in S$, $V(x)$ has a greatest element - denoted by x^0
- Let $S^0 = \{x^0 : x \in S\}$. Then S^0 is an inverse subsemigroup of S and for all $x \in S$, $|S^0 \cap V(x)| = 1$

Inverse Transversals

S^0 is an *inverse transversal* of S if for all $x \in S$ there exists a unique $x^0 \in V(x) \cap S^0$

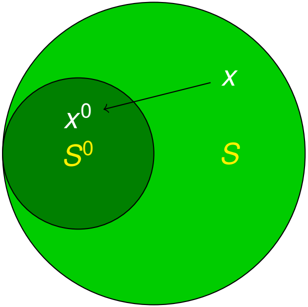
$x^{00} = (x^0)^0$

$x^{000} = x^0$

$x = (xx^0)x^{00}(x^0x) = e_x x^{00} f_x$

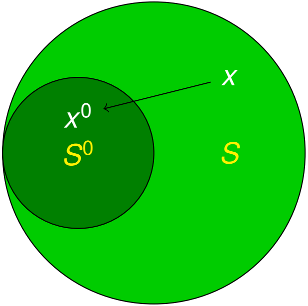
$e_x \,\mathcal{L}\, x^{00}x^0 \,\mathcal{R}\, x^{00}$

$(x^0y)^0 = y^0x^{00}$



Generalisations

An *associate* of x is an element $x' \in S$ with $xx'x = x$. S^0 is an *associate transversal* of S if for all $x \in S$ there exists a unique $x^0 \in A(x) \cap S^0$ where $A(x)$ is the set of all associates of x.



In Semigroup Forum (2009) 79, 101–118, Billhardt, Giraldes, Marques-Smith, Mendes Martins consider the situation where x^0 is the *least associate* with respect to the natural partial order on S.

Generalizations

Let $V_{S^0}(x) = V(x) \cap S^0$.

S^0 is an *orthodox transversal* of S if

1. for all $x \in S$, $V_{S^0}(x) \neq \emptyset$
2. if $a, b \in S$ and $\{a, b\} \cap S^0 \neq \emptyset$ then $V_{S^0}(a)V_{S^0}(b) \subseteq V_{S^0}(ba)$.

Easy to check that S^0 is necessarily an orthodox subsemigroup of S.

Adequate transversals

- Define a left congruence on S by

$$\mathcal{R}^* = \{(a,b) \in S \times S \mid xa = ya \text{ iff } xb = yb \text{ for all } x, y \in S^1\}$$

and a right congruence by

$$\mathcal{L}^* = \{(a,b) \in S \times S \mid ax = ay \text{ iff } bx = by \text{ for all } x, y \in S^1\}$$

- We say that a semigroup is *abundant* if each $\mathcal{R}^*-$class and each $\mathcal{L}^*-$class contains an idempotent
- An abundant semigroup in which the idempotents commute is called *adequate*

Adequate transversals

Lemma

A semigroup S is adequate if and only if each $\mathcal{L}^*-$class and each $\mathcal{R}^*-$class contain a unique idempotent and the subsemigroup generated by $E(S)$ is regular.

If S is adequate and $a \in S$ denote by a^* the unique idempotent in L_a^* and by a^+ the unique idempotent in R_a^*.

Lemma

If S is an adequate semigroup then for all $a, b \in S$, $(ab)^* = (a^*b)^*$ and $(ab)^+ = (ab^+)^+$.

Adequate transversals

- $U \subseteq S$ abundant subsemigroups - U is a $*-$*subsemigroup* of S if

$$\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U), \mathcal{R}^*(U) = \mathcal{R}^*(S) \cap (U \times U)$$

- Let S^0 be an adequate $*-$subsemigroup of the abundant semigroup S. S^0 is an *adequate transversal* of S if for each $x \in S$ there is a unique $\overline{x} \in S^0$ and $e, f \in E$ such that

$$x = e\overline{x}f \text{ and such that } e \,\mathcal{L}\, \overline{x}^+ \text{ and } f \,\mathcal{R}\, \overline{x}^*.$$

e and f are uniquely determined by x - denoted by e_x,and f_x and $E(S^0)$ by E^0.

Adequate transversals

Adequate transversals were first introduced by El-Qallali in the early 90s and might have been inspired by earlier joint work with Fountain on quasi-adequate semigroups.

Adequate transversals

If S is regular and S^0 is an inverse transversal, then S^0 is an adequate transversal and

- $\overline{x} = x^{00}$;
- $e_x = xx^0$;
- $f_x = x^0x$;
- $\overline{x}^+ = x^{00}x^0 \mathrel{\mathcal{R}} x^{00}$;
- $\overline{x}^* = x^0x^{00} \mathrel{\mathcal{L}} x^{00}$.

Adequate transversals

A non-regular based on one originally given by El-Qallali:

- S^0 is an adequate transversal of an abundant semigroup S
- M is a cancellative monoid with identity 1
- $M \times S^0$ is an adequate transversal of the abundant semigroup $M \times S$
- In fact $E(M \times S) = \{1\} \times E(S)$ and $E(M \times S^0) = \{1\} \times S^0$
- Moreover $\overline{(m, a)} = (m, \overline{a})$, $e_{(m,a)} = (1, e_a)$ and $f_{(m,a)} = (1, f_a)$.

Adequate transversals

Lemma

Let S be an abundant semigroup with an adequate transversal S^0. Then for all $x \in S$

1. $e_x \,\mathcal{R}^*\, x$ *and* $f_x \,\mathcal{L}^*\, x$,
2. *if* $x \in S^0$ *then* $e_x = x^+ \in E^0, \overline{x} = x, f_x = x^* \in E^0$,
3. *if* $x \in E^0$ *then* $e_x = \overline{x} = f_x = x$,
4. $e_{\overline{x}} \,\mathcal{L}\, e_x$ *and hence* $e_{\overline{x}} e_x = e_{\overline{x}}$ *and* $e_x e_{\overline{x}} = e_x$,
5. $f_{\overline{x}} \,\mathcal{R}\, f_x$ *and hence* $f_{\overline{x}} f_x = f_x$ *and* $f_x f_{\overline{x}} = f_{\overline{x}}$.

Adequate transversal

$$
\begin{aligned}
sx = tx \quad &\Rightarrow \quad se_x\overline{x}f_x\overline{x}^* = te_x\overline{x}f_x\overline{x}^* \\
&\Rightarrow \quad se_x\overline{x} = te_x\overline{x} \\
&\Rightarrow \quad se_x\overline{x}^+ = te_x\overline{x}^+ \\
&\Rightarrow \quad se_x = fe_x
\end{aligned}
$$

Adequate transversals

$$I = \{e_x : x \in S\}, \quad \Lambda = \{f_x : x \in S\}$$

Lemma

Let S^0 be an adequate transversal of an abundant semigroup S and let $x, y \in S$. Then

1. $x \, \mathcal{R}^* \, y$ *if and only if* $e_x = e_y$,
2. $x \, \mathcal{L}^* \, y$ *if and only if* $f_x = f_y$.

Hence there are bijections $I \to S/\mathcal{R}^*$ and $\Lambda \to S/\mathcal{L}^*$.

$|R_x^* \cap I| = 1$ and that $|L_x^* \cap \Lambda| = 1$.

Adequate transversals

Suppose now that $x \in Reg(S)$, the set of regular elements of S. Using the fact that $x \mathcal{R} e_x$ and $x \mathcal{L} f_x$ then there exists a unique $x^0 \in V(x)$ with $xx^0 = e_x$ and $x^0 x = f_x$.

Theorem

If $x \in Reg(S)$ then $|V(x) \cap S^0| = 1$. Moreover $x^0 \in S^0$, $\overline{x} = x^{00}$ and $x^0 = x^{000}$. Also,

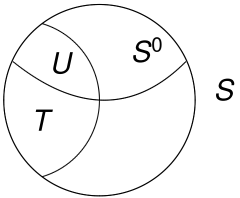
$$I = \{x \in Reg(S) : x = xx^0\} = \{xx^0 : x \in Reg(S)\}$$

and

$$\Lambda = \{x \in Reg(S) : x = x^0 x\} = \{x^0 x : x \in Reg(S)\}.$$

Adequate transversals

Let $T = Reg(S)$, let $U = T \cap S^0$ and suppose that T is a subsemigroup of S.
U is an inverse transveral of the regular semigroup T



Theorem

If T is a subsemigroup of S then I is a left regular subband of S and Λ is a right regular subband of S.

Quasi-adequate

- A semigroup is said to be *quasi-adequate* if it is abundant and its idempotents form a subsemigroup.
- It was shown by El-Qallali and Fountain that in this case the set T of regular elements is an orthodox subsemigroup of S.
- So we see that $U = T \cap S^0$ is an inverse transversal of T and I and Λ are subbands of S.

Quasi-adequate

Proposition

Let S^0 be an adequate transversal of an abundant semigroup S. Then the following are equivalent:

1. *S is quasi-adequate;*
2. $(\forall x, y \in Reg(S)), (xy)^0 = y^0 x^0$;
3. $(\forall i \in I)(\forall l \in \Lambda), (li)^0 = i^0 l^0$;
4. $I\Lambda = E(S)$.

Quasi-adequate

- Let S be a quasi-adequate semigroup with an adequate transversal S^0 and suppose that $x, y \in Reg(S)$. Then $\overline{xy} = \overline{x}\,\overline{y}$.
- Let S be an orthodox semigroup with an adequate (and hence inverse) transversal S^0. Then for all $x, y \in S$, $\overline{xy} = \overline{x}\,\overline{y}$.

  We say that S^0 is a *quasi-ideal* of S if $S^0SS^0 \subseteq S^0$ or equivalently if $\Lambda I \subseteq S^0$. These transversals have been the subject of a great deal of study in both the inverse and adequate cases.
- Let S be an abundant semigroup with a quasi-ideal adequate transversal S^0. S is quasi-adequate if and only if for all $x, y \in S$, $\overline{xy} = \overline{x}\,\overline{y}$.

Quasi-adequate

- S quasi-adequate semigroup, band of idempotents E
- for $e \in E$, let $E(e)$ denote the $\mathcal{J}-$class of e in E
- for $a \in S$, let a^+ denote a typical element of $R_a^*(S) \cap E$ and let a^* denote a typical element of $L_a^*(S) \cap E$.
- Define a relation δ on S by

$$\delta = \{(a, b) \in S \times S : b = eaf, \text{ for some } e \in E(a^+), f \in E(a^*)\}.$$

Fountain showed that δ is an equivalence relation and is contained in any adequate congruence ρ on S.

Quasi-adequate

$\phi : S \to T$ is called *good* if for all $a, b \in S$, $a\, \mathcal{R}^*(S)\, b$ implies $a\phi\, \mathcal{R}^*(T)\, b\phi$ and $a\, \mathcal{L}^*(S)\, b$ implies $a\phi\, \mathcal{L}^*(T)\, b\phi$.
A congruence ρ is called *good* if the natural homomorphism $\rho^\natural : S \to S/\rho$ is good.

Lemma (Fountain)

If δ is a congruence then δ is the minimum adequate good congruence on S.

Quasi-adequate

Proposition (El-Qallali)

If S is a quasi-adequate semigroup with an adequate transversal S^0 then the following are equivalent

1. δ *is a congruence on* S,
2. $\delta = \{(a, b) \in S \times S : \overline{a} = \overline{b}\}$,
3. *for all* $x, y \in S$, $\overline{xy} = \overline{x}\,\overline{y}$.

Moreover in this case $S/\delta \cong S^0$.

Consequently, we shall say that an adequate transversal S^0 of a quasi-adequate semigroup S is *good* if $\overline{xy} = \overline{x}\,\overline{y}$ for all $x, y \in S$.

Structure Theorems

Theorem

Let S be a quasi-adequate semigroup with a good adequate transversal S^0. Then

1. $\overline{xy} = \overline{\overline{x} f_x e_y \overline{y}}$
2. $e_{xy} = e_x e_{\overline{x} f_x e_y \overline{y}}$
3. $f_{xy} = f_{\overline{x} f_x e_y \overline{y}} f_y$.

$$(e_x \overline{x} f_x)(e_y \overline{y} f_y) = (e_x e_{\overline{x} f_x e_y \overline{y}}) \left(\overline{\overline{x} f_x e_y \overline{y}} \right) (f_{\overline{x} f_x e_y \overline{y}} f_y).$$

Structure Theorems

$I = \cup_{x \in E^0} L_x$ and I is a semilattice E^0 of the left zero semigroups L_x.

Theorem

S^0 adequate semigroup with semilattice E^0, $I = \cup_{x \in E^0} L_x$ left regular band, $\Lambda = \cup_{x \in E^0} R_x$ right regular band, common semilattice transversal E^0.
$\forall x, y \in S^0, \exists \alpha_{x,y} : R_{x^*} \times L_{y^+} \to L_{(xy)^+}, \beta_{x,y} : R_{x^*} \times L_{y^+} \to R_{(xy)^*}$ satisfying:

1. *if $f \in R_{x^*}, g \in L_{y^+}, h \in R_{y^*}, k \in L_{z^+}$ then*
$$(f, g)\alpha_{x,y} \left((f, g)\beta_{x,y} h, k\right) \alpha_{xy,z} = \left(f, g(h, k)\alpha_{y,z}\right) \alpha_{x,yz}$$
$$\left(f, g(h, k)\alpha_{y,z}\right) \beta_{x,yz}(h, k)\beta_{y,z} = \left((f, g)\beta_{x,y} h, k\right) \beta_{xy,z},$$
2. $(x^*, y^+)\alpha_{x,y} = (xy)^+, (x^*, y^+)\beta_{x,y} = (xy)^*,$

Structure Theorems

Theorem

3. *if*
$x, x_1, x_2 \in S^0$, $e_1 \in L_{x_1^+}$, $f_1 \in R_{x_1^*}$, $e_2 \in L_{x_2^+}$, $f_2 \in R_{x_2^*}$, $e \in L_{x^+}$
and if
   - $e_1(f_1, e)\alpha_{x_1,x} = e_2(f_2, e)\alpha_{x_2,x}$,
   - $x_1 x = x_2 x$
   - $(f_1, e)\beta_{x_1,x} x^* = (f_2, e)\beta_{x_2,x} x^*$

   then
   - $e_1(f_1, e)\alpha_{x_1,x^+} = e_2(f_2, e)\alpha_{x_2,x^+}$,
   - $x_1 x^+ = x_2 x^+$
   - $(f_1, e)\beta_{x_1,x^+} = (f_2, e)\beta_{x_2,x^+}$.

Structure Theorems

Theorem

4. *if*
$x, x_1, x_2 \in S^0$, $e_1 \in L_{x_1^+}$, $f_1 \in R_{x_1^*}$, $e_2 \in L_{x_2^+}$, $f_2 \in R_{x_2^*}$, $f \in R_{x^*}$
and if
   - $x^+(f, e_1)\alpha_{x,x_1} = x^+(f, e_2)\alpha_{x,x_2}$,
   - $xx_1 = xx_2$
   - $(f, e_1)\beta_{x,x_1} f_1 = (f, e_2)\beta_{x,x_2} f_2$

   then
   - $(f, e_1)\alpha_{x^*,x_1} = (f, e_2)\alpha_{x^*,x_2}$,
   - $x^* x_1 = x^* x_2$
   - $(f, e_1)\beta_{x^*,x_1} f_1 = (f, e_2)\beta_{x^*,x_2} f_2$

Structure Theorems

Theorem

Define a multiplication on the set

$$W = \{(e, x, f) \in I \times S^0 \times \Lambda : e \in L_{x^+}, f \in R_{x^*}\}$$

by

$$(e, x, f)(g, y, h) = (e(f, g)\alpha_{x,y}, xy, (f, g)\beta_{x,y}h).$$

Then W is a quasi-adequate semigroup with a good adequate transversal isomorphic to S^0.

Moreover every quasi-adequate semigroup S, with a good adequate transversal can be constructed in this way.

$(a, b)\alpha_{x,y} = e_{xaby}$ and $(a, b)\beta_{x,y} = f_{xaby}$

Structure Theorems

Corollary

Let S^0 be an adequate semigroup with semilattice of idempotents E^0 and let $I = \cup_{x \in E^0} L_x$ be a left normal band and $\Lambda = \cup_{x \in E^0} R_x$ be a right normal band with a common semilattice transversal E^0. Let

$$W = \{(e, x, f) \in I \times S^0 \times \Lambda : e \in L_{x^+}, f \in R_{x^*}\}$$

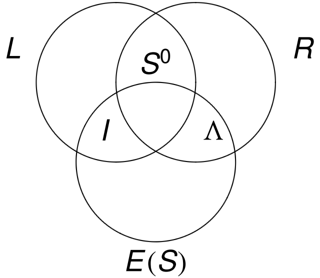
and define a multiplication on W by

$$(e, x, f)(g, y, h) = (e(xy)^+, xy, (xy)^* h).$$

Then W is a quasi-adequate semigroup with a quasi-ideal, good adequate transversal isomorphic to S^0. Conversely every such transversal can be constructed in this way.

Structure Theorems

$$R = \{x \in S : e_x = e_{\overline{x}}\},\ L = \{x \in S : f_x = f_{\overline{x}}\}$$



It can be shown that

$$R = \{x \in S : x = \overline{x}f_x\} = \{\overline{x}f_x : x \in S\},$$

$$L = \{x \in S : x = e_x\overline{x}\} = \{e_x\overline{x} : x \in S\}.$$

Structure Theorems

An abundant semigroup is *left (resp. right) adequate* if every $\mathcal{R}^*-$class (resp. $\mathcal{L}^*-$class) contains a unique idempotent.

Corollary

Let L be a left adequate semigroup and R a right adequate semigroup with a common quasi-ideal adequate transversal S^0. Construct the spined product

$$L| \times |R = \{(x, a) \in L \times R : \overline{x} = \overline{a}\}$$

and define a multiplication on $L| \times |R$ by

$$(x, a)(y, b) = (x\overline{y}, \overline{a}b) = (x\overline{b}, \overline{x}b).$$

Then $L| \times |R$ is a quasi-adequate semigroup with a good, quasi-ideal adequate transversal isomorphic to S^0. Moreover every such transversal can be constructed in this way.

Structure Theorems

- Let S be a left adequate semigroup. Since $\Lambda = E^0$ then if S is also quasi-adequate and S^0 is good then we must have $R_{x^*} = \{x^*\}$ and so $(f, e)\beta_{x,y} = (xy)^*$.
- I a left regular band with a semilattice transversal E^0. Define on I a left S^0-action $S^0 \times I \to I$, $(x, e) \mapsto x * e$ and which is *distributive* over the multiplication on I, i.e $(xy) * e = x * (y * e)$ and $x * (ef) = (x * e)(x * f)$.
- Construct the semidirect product of S^0 by I as $I * S^0 = \{(e, x) \in I \times S^0\}$ with multiplication given by

$$(e, x)(g, y) = (e(x * g), xy)$$

and it is an easy matter to check that $I * S^0$ is a semigroup.

Structure Theorems

We say that a left adequate semigroup S is *left ample* (formerly called *left type-A*) if for all $a \in S$, $e \in E(S)$, $(ae) = (ae)^+a$.

Theorem

Let S^0 be a left ample, adequate semigroup with semilattice E^0 and let $I = \cup_{x \in E^0} L_x$ be a left regular band with a semilattice transversal E^0. Suppose there is a left S^0−action $S^0 \times I \to I$, $(x, e) \mapsto x * e$ which is distributive over I satisfying:

1. *for all $x, y \in S^0$, $x * y^+ = (xy)^+$,*
2. *if $x, x_1, x_2 \in S^0$, $e_1 \in L_{x_1^+}$, $e_2 \in L_{x_2^+}$ and if*

$$x^+(x * e_1) = x^+(x * e_2),\ \ xx_1 = xx_2$$

then

$$x^* * e_1 = x^* * e_2,\ \ x^*x_1 = x^*x_2$$

Structure Theorems

Theorem

Define a multiplication on the set

$$W = \{(e, x) \in I \times S^0 : e \in L_{x^+}\}$$

by

$$(e, x)(g, y) = (e(x * g), xy).$$

Then W is a left adequate, quasi-adequate semigroup with a good, left ample, adequate transversal isomorphic to S^0.

Moreover every left adequate, quasi-adequate semigroup S with a left ample, good adequate transversal can be constructed in this way.

Regular semigroups

Corollary

Let S^0 be an inverse semigroup with semilattice of idempotents E^0 and let I be a left regular band with a semilattice transversal isomorphic to E^0. Suppose we have a left action of S^0 on I, $(x, e) \mapsto x * e$ and which is distributive over the multiplication on I satisfying

1. *for all $x, y \in S^0$, $x * (yy^{-1}) = (xy)(xy)^{-1}$;*
2. *for all $x \in S^0$, $e \in I$, $(xx^{-1}) * e = (xx^{-1})e$.*

Define a multiplication on $W = \{(e, x) \in I \times S^0 : e \in L_{xx^{-1}}\}$ by $(e, x)(g, y) = (e(x * g), xy)$.
Then W is a left inverse semigroup with an inverse transversal isomorphic to S^0. Moreover every left inverse semigroup S with an inverse transversal can be constructed in this way.

Quasi-ideals

Theorem

Let S^0 be adequate with semilattice of idempotents E^0. Let L be left adequate and R right adequate and suppose that S^0 is a common quasi-ideal adequate transversal of both. Let $* : R \times L \to S^0$ be a map such that

1. *for all $y, z \in L$, $a, b \in R$ with $\overline{y} = \overline{b}$, $(a * y)f_b * z = a * e_y(b * z)$;*
2. *if $a \in S^0$ or $x \in S^0$ then $a * x = ax$.*

Let $T = \{(x, a) \in L \times R : \overline{x} = \overline{a}\}$ and define

$$(x, a)(y, b) = (e_x(a * y), (a * y)f_b).$$

Then T is an abundant semigroup with a quasi-ideal adequate transversal T^0 with $T^0 \cong S^0$. Moreover every quasi-ideal adequate transversal can be constructed in this way.