Relational depth of semigroups with chain-like $\mathcal{J}$-classes

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Green's $\mathcal{J}$-relation

- In a semigroup S,

$$(x, y) \in \mathcal{J} \iff S^1 x S^1 = S^1 y S^1.$$

- The $\mathcal{J}$-classes of S are of the form

$$J_x = \{x' \in S : (x', x) \in \mathcal{J}\}.$$

- Some semigroups have $\mathcal{J}$-classes that form a chain ($J_x \leq J_y \iff S^1 x S^1 \subseteq S^1 y S^1$).

Semigroups with chain-like $\mathcal{J}$-classes

> Transformation Semigroups

- PT_n – all partial transformations on $[n]$

  e.g. $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & - & 2 & 5 & 6 \end{pmatrix} \in PT_6$

- I_n – all partial bijections on $[n]$

  e.g. $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & - & 5 & 6 \end{pmatrix} \in I_6$

- T_n – all functions from $[n]$ into $[n]$

  e.g. $\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \end{pmatrix} \in T_6$

> Partition monoid, Brauer monoid. Temperley-Lieb monoid

> Matrix semigroups

$\mathcal{J}$-classes and ranks

> For $S \in \{T_n, I_n, PT_n\}$, $\alpha \mathcal{J} \beta \iff \operatorname{rank}\alpha = \operatorname{rank}\beta$.

- The $\mathcal{J}$-classes of S are of the form

$$J_i := \{\alpha \in S : \operatorname{rank}\alpha = i\}$$

- The $\mathcal{J}$-classes of S form a chain,

$$J_\varepsilon < \cdots < J_n \quad (\varepsilon \in \{0,1\}).$$

$\mathcal{J}$-classes of the ideals of T_n, $\mathcal{I}_n$, PT_n

- A subset I of semigroup S is an ideal of S if for all $s \in S$, $i \in I$, si, $is \in I$.

- The ideals of T_n, $\mathcal{I}_n$, PT_n are of the form

$$I_m := \{ \alpha \in S : \text{rank } \alpha \leq m \} \quad (\varepsilon \leq m \leq n).$$

The $\mathcal{J}$-classes of I_m are $J_\varepsilon < \cdots < J_m$.

Semigroup Presentations

- $\langle A \mid R \rangle$ is a presentation for the semigroup S if $S \cong A^+/R^\#$,

  alphabet

  $\subseteq A^+ \times A^+$

  where $R^\#$ is the smallest congruence on A^+ that contains R.

- The *Cayley table presentation* for a semigroup S is

$$C = \langle x_s \ (s \in S) \mid x_s x_t = x_{st} \ (s,t \in S) \rangle.$$

Semigroup presentation for S_n

Proposition (Moore, 1897)

The presentation

$$\langle a, b \mid a^2 = b^n = (ba)^{n-1} = (ab^{n-1}ab)^3$$
$$= (ab^{n-j}ab^j)^2 = 1 \quad (2 \le j \le n-2) \rangle$$

defines S_n in terms of generators $a = (1\ 2)$ and $b = (1\ 2\ \dots\ n)$.

Presentation for T_n

Proposition (Aĭzenštat, 1958)

Suppose that $\langle a, b \mid R \rangle$ is a (semigroup) presentation for S_n, where a represents $(1\,2)$, b represents $(1\,2\,\cdots\,n)$. Let t represent the transformation

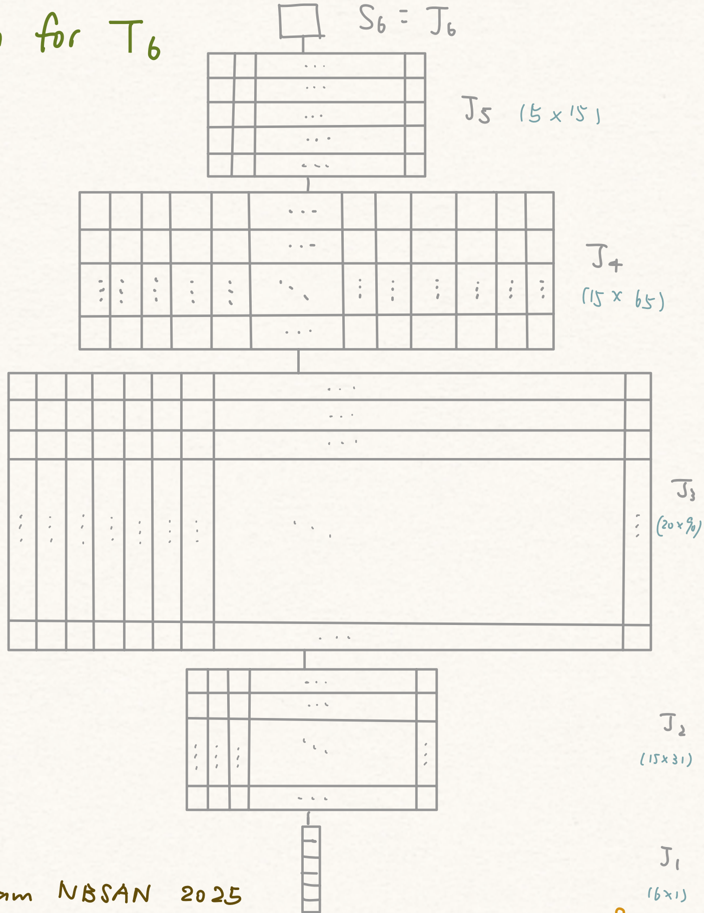
$$\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 3 & \cdots & n \end{pmatrix} \in T_n.$$

Then the presentation

$$\langle a, b, t \mid R,\ at = b^{n-2}ab^2tb^{n-2}ab^2 = bab^{n-1}abtb^{n-1}abab^{n-1} = (tbab^{n-1})^2 = t,\ (b^{n-1}abt)^2 = tb^{n-1}abt = (tb^{n-1}ab)^2,\ (tbab^{n-2}ab)^2 = (bab^{n-2}ata)^2 \rangle$$

defines T_n.

Ayzenštat's presentation for T_6



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Aizenštat's presentation for T_6

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}$$

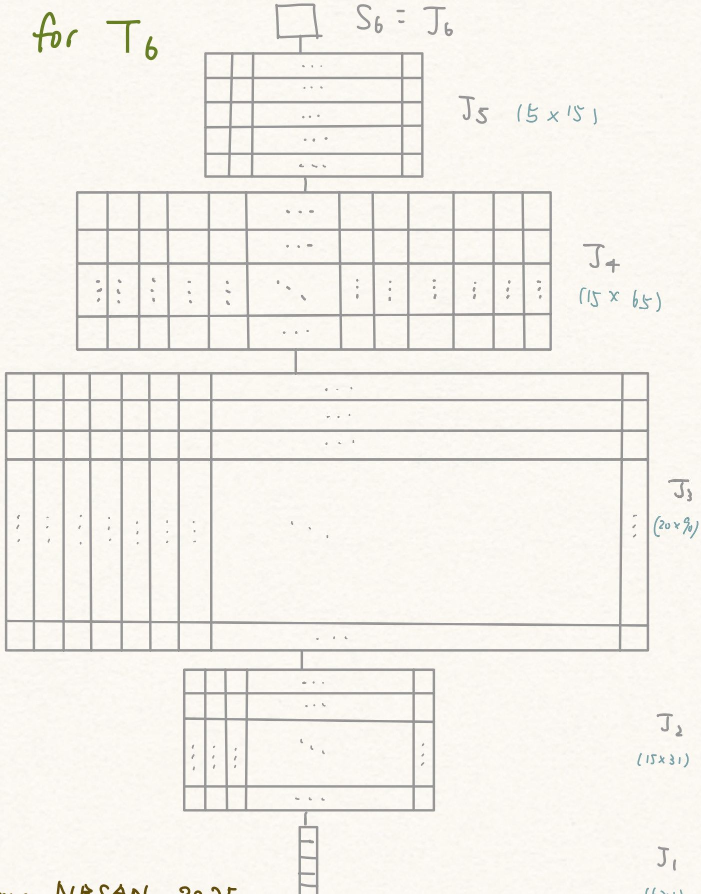
$$t = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 3 & 4 & 5 & 6 \end{pmatrix}$$

Relations in Aizenštat's presentation are:

R, $at = \cdots = t$,

$(b^{n-1}abt)^2 = \cdots = (tb^{n-1}ab)^2$,

$(tbab^{n-2}ab)^2 = (bab^{n-2}ata)^2$.



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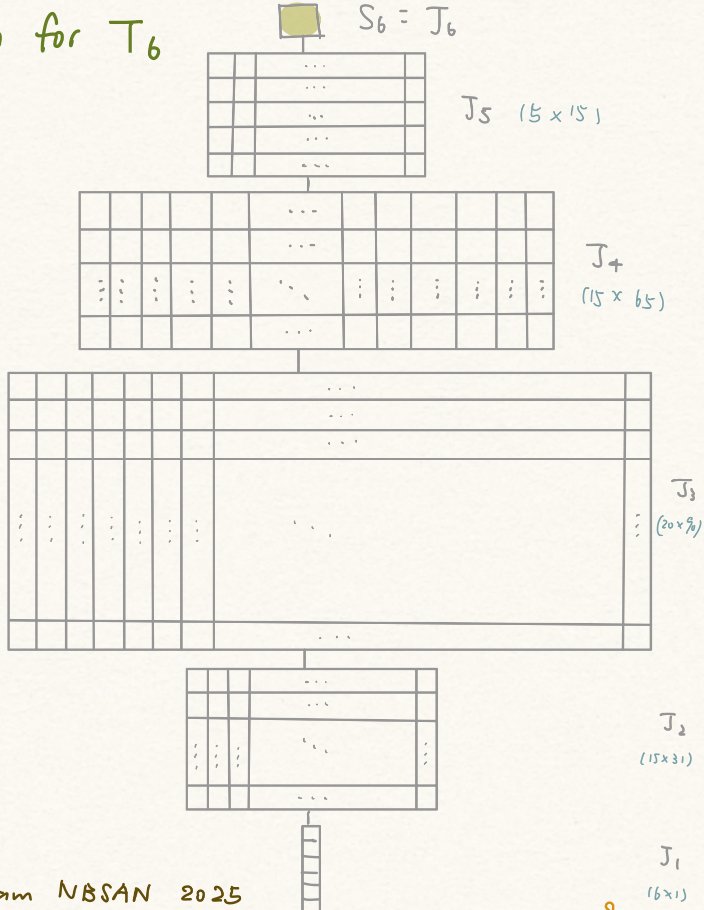

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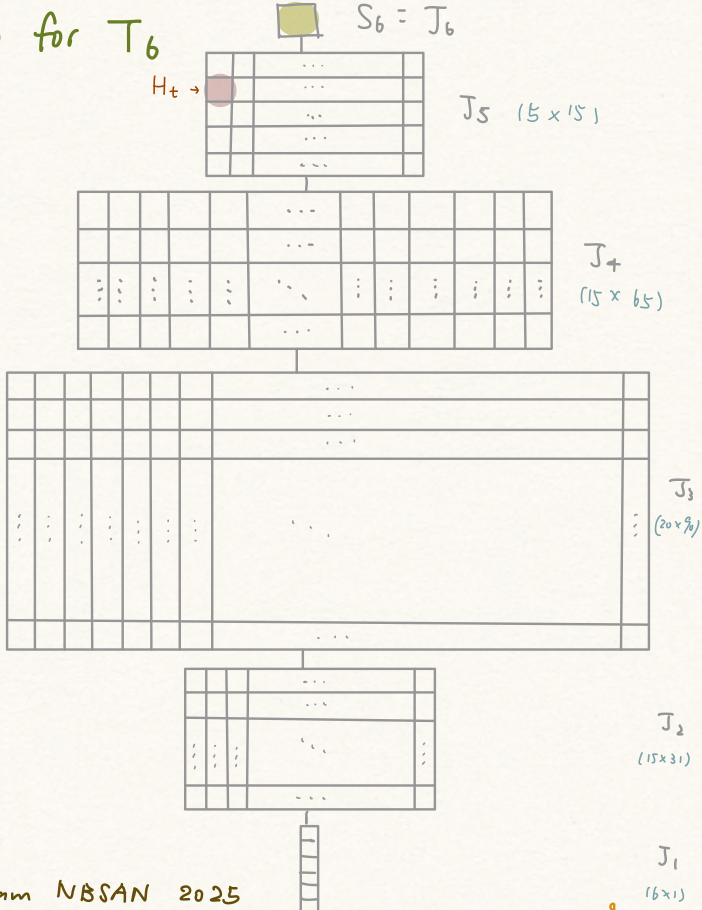

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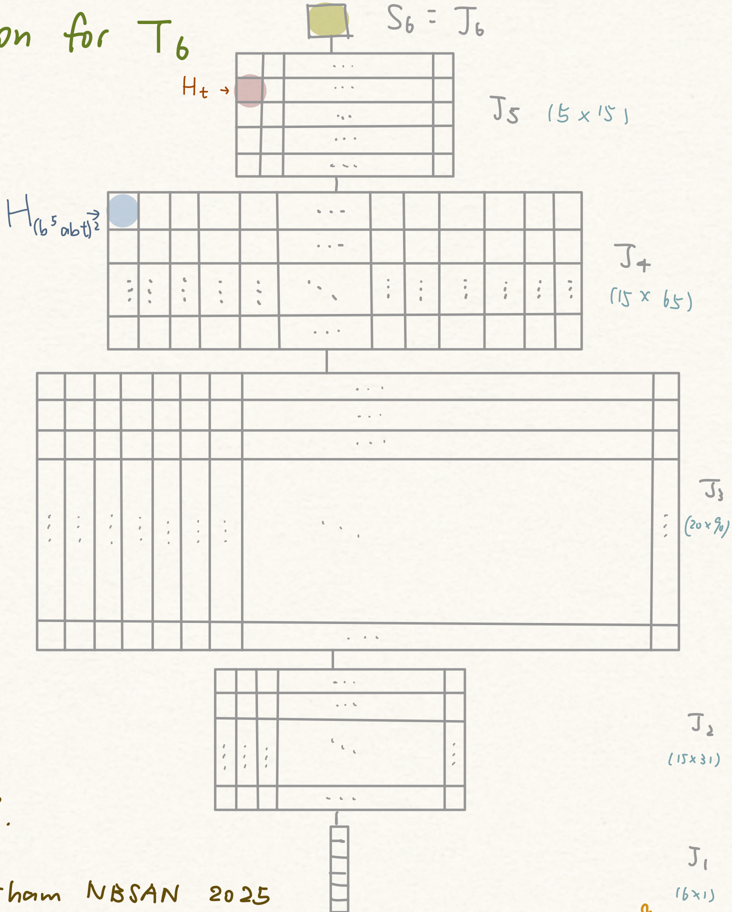

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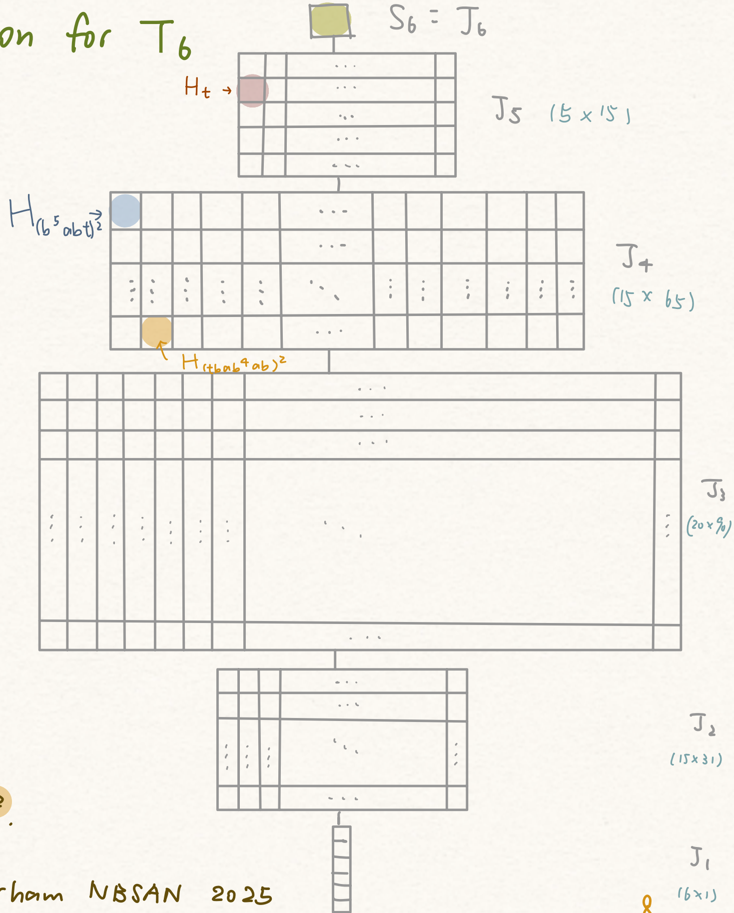

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Presentation for $T_n \setminus S_n$

Proposition (East, 2013)

Define maps α_{ij} $(1 \le i < j \le n)$ by $x\alpha_{ij} = \begin{cases} x & \text{if } 1 \le x < j \\ i & \text{if } x = j \\ x-1 & \text{if } j < x \le n, \end{cases}$ β_i $(1 \le i \le n-1)$ by

$x\beta_i = \begin{cases} x & \text{if } x < i \text{ or } x = n \\ x+1 & \text{if } i \le x < n, \end{cases}$ σ_i $(1 \le i \le n-2)$ by $x\sigma_i = \begin{cases} x & \text{if } x \ne i, i+1, n \\ i+1 & \text{if } x = i \\ i & \text{if } x = i+1 \\ (n-1)\sigma_i & \text{if } x = n \end{cases}.$

Define an alpabet $Y = A \cup B \cup S$, where

$A = \{a_{ij} \mid 1 \le i < j \le n\}$, $B = \{b_i \mid 1 \le i \le n-1\}$. $S = \{s_i \mid 1 \le i \le n-2\}$.

Let Q be the set of relations

$$a_{kl}a_{in} = a_{kl} \quad \text{for all } i, k, l \tag{A1}$$

$$a_{jk}a_{ij} = a_{ik}a_{ij} = a_{ij}a_{i,k-1} \quad \text{if } i < j < k \tag{A2}$$

$$a_{kl}a_{ij} = \begin{cases} a_{ij}a_{k-1,l-1} & \text{if } i < j < k < l & \text{(A3)} \\ a_{ij}a_{k,l-1} & \text{if } i < k < j < l & \text{(A4)} \\ a_{i,j+1}a_{kl} & \text{if } i < k < l \le j < n; & \text{(A5)} \end{cases}$$

$$b_jb_i = b_ib_{j+1} \quad \text{if } 1 \le i \le j \le n-2 \tag{B1}$$

$$b_{n-1}b_i = b_i \quad \text{for all } i; \tag{B2}$$

$$s_ia_{n-1,n} = a_{n-1,n}s_i = s_i \quad \text{for all } i \tag{S1}$$

$$s_i^2 = a_{n-1,n} \quad \text{for all } i \tag{S2}$$

$$s_is_j = s_js_i \quad \text{if } |i-j| > 1 \tag{S3}$$

$$s_is_js_i = s_js_is_j \quad \text{if } |i-j| = 1; \tag{S4}$$

$$s_ra_{ij} = \begin{cases} a_{n-1,n}a_{ij}s_r & \text{if } r \le i-2 \text{ and } j < n & \text{(SA1)} \\ a_{n-1,n}a_{i-1,j}s_r & \text{if } r = i-1 \text{ and } j < n & \text{(SA2)} \\ a_{n-1,n}a_{i+1,j}s_r & \text{if } r = i < j-1 \text{ and } j < n & \text{(SA3)} \\ a_{n-1,n}a_{ij} & \text{if } r = i = j-1 & \text{(SA4)} \\ a_{n-1,n}a_{ij}s_r & \text{if } i < r < j-1 \text{ and } j < n & \text{(SA5)} \\ a_{n-1,n}a_{i,j-1} & \text{if } i < r = j-1 & \text{(SA6)} \\ a_{n-1,n}a_{i,j+1} & \text{if } r = j & \text{(SA7)} \\ a_{n-1,n}a_{ij}s_{r-1} & \text{if } j < r & \text{(SA8)} \\ s_r & \text{if } j = n; & \text{(SA9)} \end{cases}$$

$$b_is_r = \begin{cases} a_{n-1,n}a_{n-2,n-1}s_rb_i & \text{if } r \le i-2 \text{ and } i < n-1 & \text{(BS1)} \\ s_r & \text{if } r \le i-2 \text{ and } i = n-1 & \text{(BS2)} \\ a_{n-1,n}a_{n-2,n-1}b_{i-1} & \text{if } r = i-1 \text{ and } i < n-1 & \text{(BS3)} \\ s_r & \text{if } r = i-1 \text{ and } i = n-1 & \text{(BS4)} \\ a_{n-1,n}a_{n-2,n-1}b_{i+1} & \text{if } r = i & \text{(BS5)} \\ a_{n-1,n}a_{n-2,n-1}s_{i+1} & \text{if } r = i = n-3 & \text{(BS6)} \\ a_{n-1,n}a_{n-2,n-1} & \text{if } r = i = n-2 & \text{(BS7)} \\ a_{n-1,n}a_{n-2,n-1}s_{r-1}b_i & \text{if } i < r; & \text{(BS8)} \end{cases}$$

$$b_ra_{ij} = \begin{cases} a_{n-1,n}a_{i-1,j-1}b_r & \text{if } r < i & \text{(BA1)} \\ s_{j-2}\cdots s_i & \text{if } r = i < j-1 & \text{(BA2)} \\ a_{n-1,n} & \text{if } r = i = j-1 & \text{(BA3)} \\ a_{n-1,n}a_{i,j-1}b_r & \text{if } i < r < j & \text{(BA4)} \\ a_{n-1,n} & \text{if } r = j & \text{(BA5)} \\ a_{n-1,n}a_{ij}b_{r-1} & \text{if } j < r; & \text{(BA6)} \\ a_{n-1,n}a_{ij}b_{r-1} & & \text{(BA7)} \end{cases}$$

The semigroup $T_n \setminus S_n$ has presentation $\langle Y \mid Q \rangle$.

East's presentation for $I_5 = T_6 \setminus S_6$

$$
\begin{array}{lll}
a_{kl}a_{in} = a_{kl} & \text{for all } i,k,l & \text{(A1)}\\
a_{jk}a_{ij} = a_{ik}a_{ij} = a_{ij}a_{i,k-1} & \text{if } i<j<k & \text{(A2)}\\
a_{kl}a_{ij} = \begin{cases} a_{ij}a_{k-1,l-1} \\ a_{ij}a_{k,l-1} \\ a_{i,j+1}a_{kl} \end{cases} & \begin{array}{l} \text{if } i<j<k<l \\ \text{if } i<k<j<l \\ \text{if } i<k<l\le j<n; \end{array} & \begin{array}{l} \text{(A3)} \\ \text{(A4)} \\ \text{(A5)} \end{array}\\
b_jb_i = b_ib_{j+1} & \text{if } 1\le i\le j\le n-2 & \text{(B1)}\\
b_{n-1}b_i = b_i & \text{for all } i; & \text{(B2)}\\
s_ia_{n-1,n} = a_{n-1,n}s_i = s_i & \text{for all } i & \text{(S1)}\\
s_i^2 = a_{n-1,n} & \text{for all } i & \text{(S2)}\\
s_is_j = s_js_i & \text{if } |i-j|>1 & \text{(S3)}\\
s_is_js_i = s_js_is_j & \text{if } |i-j|=1; & \text{(S4)}
\end{array}
$$

$$
s_ra_{ij} = \begin{cases}
a_{n-1,n}a_{ij}s_r & \text{if } r\le i-2 \text{ and } j<n \quad \text{(SA1)}\\
a_{n-1,n}a_{i-1,j}s_r & \text{if } r=i-1 \text{ and } j<n \quad \text{(SA2)}\\
a_{n-1,n}a_{i+1,j}s_r & \text{if } r=i<j-1 \text{ and } j<n \quad \text{(SA3)}\\
a_{n-1,n}a_{ij} & \text{if } r=i=j-1 \quad \text{(SA4)}\\
a_{n-1,n}a_{ij}s_r & \text{if } i<r<j-1 \text{ and } j<n \quad \text{(SA5)}\\
a_{n-1,n}a_{i,j-1} & \text{if } i<r=j-1 \quad \text{(SA6)}\\
a_{n-1,n}a_{i,j+1} & \text{if } r=j \quad \text{(SA7)}\\
a_{n-1,n}a_{ij}s_{r-1} & \text{if } j<r \quad \text{(SA8)}\\
s_r & \text{if } j=n; \quad \text{(SA9)}
\end{cases}
$$

$$
b_is_r = \begin{cases}
a_{n-1,n}a_{n-2,n-1}s_rb_i & \text{if } r\le i-2 \text{ and } i<n-1 \quad \text{(BS1)}\\
s_r & \text{if } r\le i-2 \text{ and } i=n-1 \quad \text{(BS2)}\\
a_{n-1,n}a_{n-2,n-1}b_{i-1} & \text{if } r=i-1 \text{ and } i<n-1 \quad \text{(BS3)}\\
s_r & \text{if } r=i-1 \text{ and } i=n-1 \quad \text{(BS4)}\\
a_{n-1,n}a_{n-2,n-1}b_{i+1} & \text{if } r=i \quad \text{(BS5)}\\
a_{n-1,n}a_{n-2,n-1}s_{i+1} & \text{if } r=i=n-3 \quad \text{(BS6)}\\
a_{n-1,n}a_{n-2,n-1} & \text{if } r=i=n-2 \quad \text{(BS7)}\\
a_{n-1,n}a_{n-2,n-1}s_{r-1}b_i & \text{if } i<r; \quad \text{(BS8)}
\end{cases}
$$

$$
b_ra_{ij} = \begin{cases}
a_{n-1,n}a_{i-1,j-1}b_r & \text{if } r<i \quad \text{(BA1)}\\
s_{j-2}\cdots s_i & \text{if } r=i<j-1 \quad \text{(BA2)}\\
a_{n-1,n} & \text{if } r=i=j-1 \quad \text{(BA3)}\\
a_{n-1,n}a_{i,j-1}b_r & \text{if } i<r<j \quad \text{(BA4)}\\
a_{n-1,n} & \text{if } r=j \quad \text{(BA5)}\\
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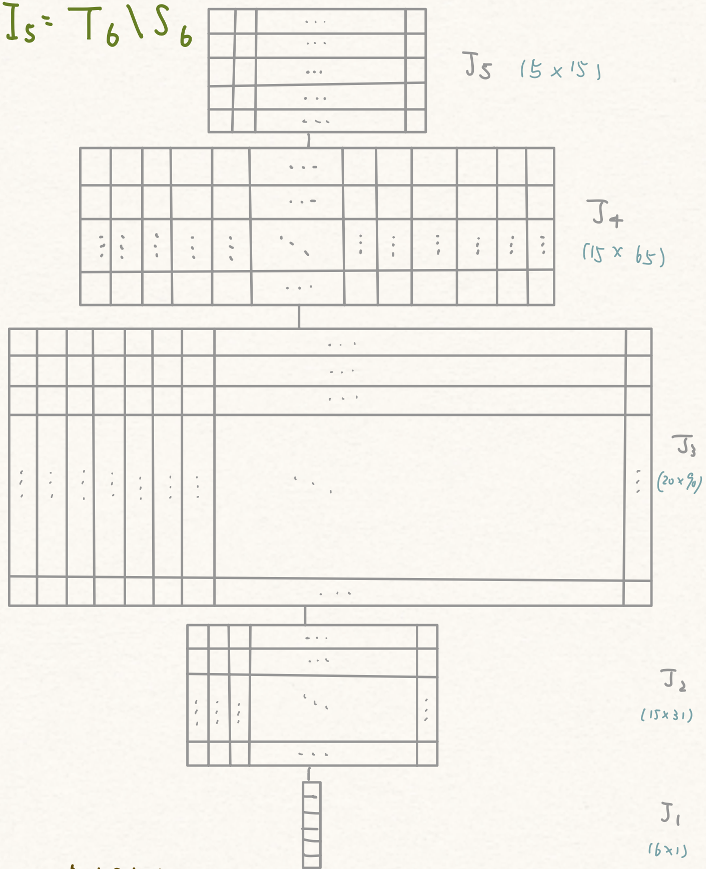
\end{cases}
$$

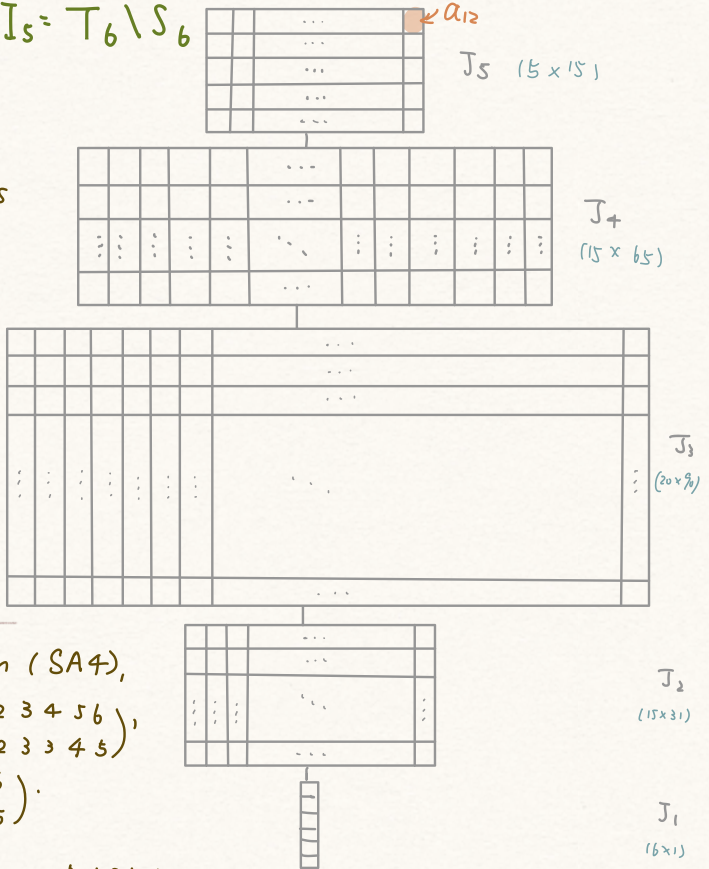


East's presentation for $I_5 = T_6 \setminus S_6$

- $a_{12}a_{16} = a_{12}$ is a relation in (A1), where a_{12} represents $\alpha_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 2 & 3 & 4 & 5 \end{pmatrix}$.

- $s_3 s_1 = s_1 s_3$ is a relation in (S3), where s_3 represents $\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 3 & 5 & 5 \end{pmatrix}$. and s_1 represents $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 5 \end{pmatrix}$.

- $s_3 a_{34} = a_{56} a_{34}$ is a relation in (SA4), where a_{34} represents $\alpha_{34} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 3 & 4 & 5 \end{pmatrix}$, a_{56} represents $\alpha_{56} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 5 \end{pmatrix}$.



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  represents $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 5 \end{pmatrix}$.

- $s_3 a_{34} = a_{56} a_{34}$ is a relation in (SA4),

  where a_{34} represents $\alpha_{34} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 3 & 4 & 5 \end{pmatrix}$,

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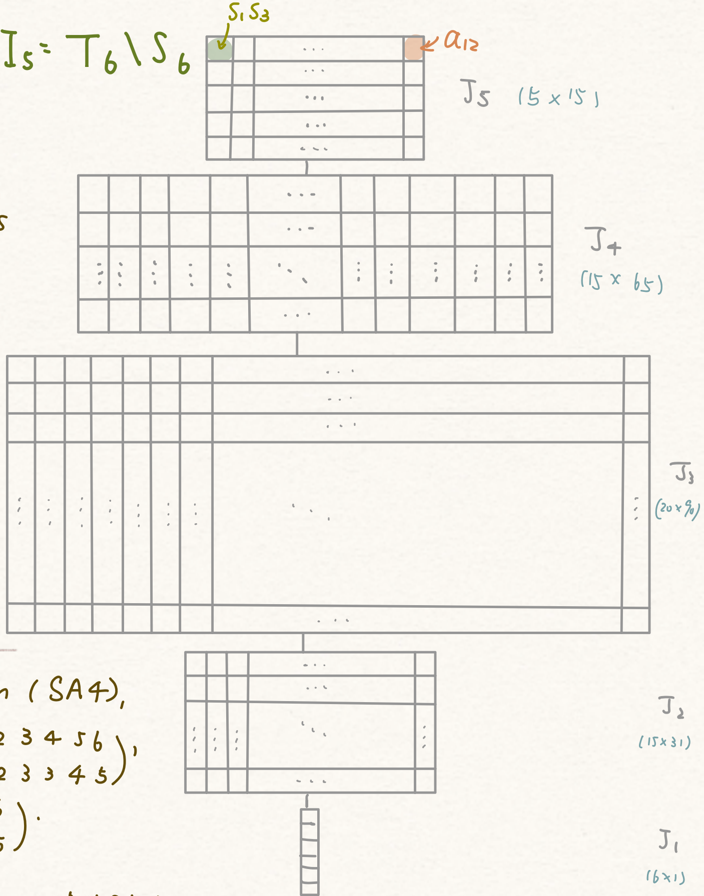

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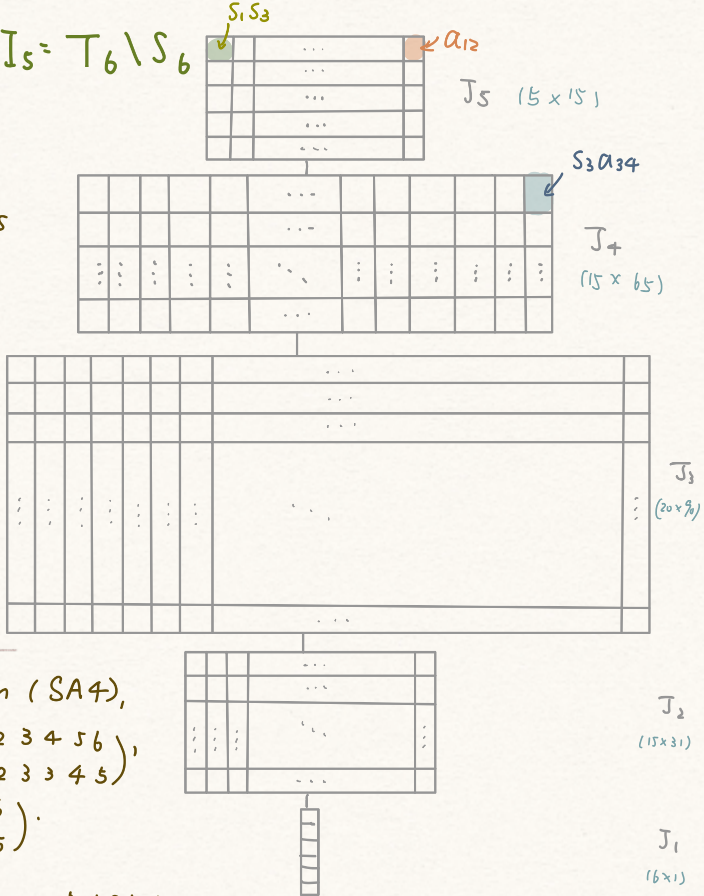

- $a_{12} a_{16} = a_{12}$ is a relation in (A1), where a_{12} represents $\alpha_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 2 & 3 & 4 & 5 \end{pmatrix}$.

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- $s_3 a_{34} = a_{56} a_{34}$ is a relation in (SA4), where a_{34} represents $\alpha_{34} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 3 & 4 & 5 \end{pmatrix}$, a_{56} represents $\alpha_{56} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 5 \end{pmatrix}$.

Presentation for $\mathcal{I}_n$

Proposition (Meakin, 1993)

Let $\langle a_1, \dots, a_{n-1} \mid R \rangle$ be a (semigroup) presentation for S_n, where a_i represents the transposition $\alpha_i = (i\ \ i{+}1)$ for $i = 1, \dots, n-1$. Let t represent the partial bijection

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-1 & - \end{pmatrix} \in \mathcal{I}_n .$$

Then the presentation

$$\langle a_1, \dots, a_{n-1}, t \mid R,\ t^2 = t,\ t a_{n-1} t = t a_{n-1} t a_{n-1} = a_{n-1} t a_{n-1} t,$$
$$t a_i = a_i t \ (1 \le i \le n-2) \rangle$$

defines $\mathcal{I}_n$.

Meakin's presentation for I_6

For $1 \le i \le 5$, a_i represents

$\alpha_i = (i \quad i+1)$, t represents

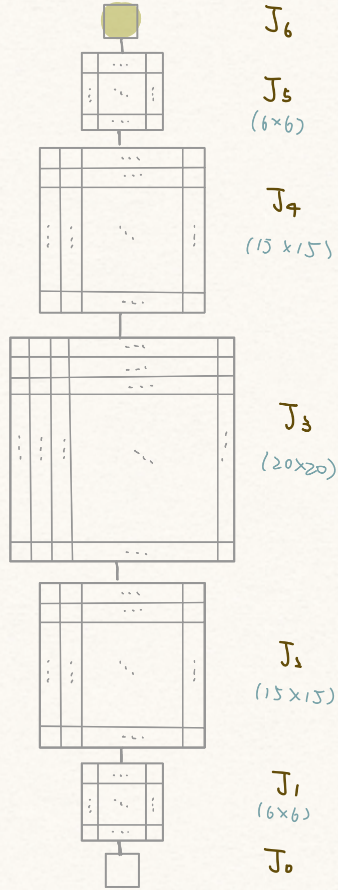
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & - \end{pmatrix}.$$

The relations in the presentation are:

R, $t^2 = t$,

$ta_5t = \cdots = a_5ta_5t$,

$ta_i = a_it \ (1 \le i \le n-2)$.



Meakin's presentation for I_6

For $1 \leq i \leq 5$, a_i represents

$\alpha_i = (i \quad i+1)$, t represents

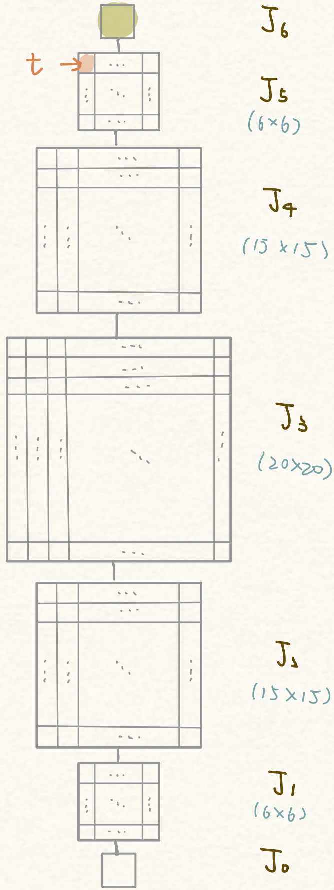
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & - \end{pmatrix}.$$

The relations in the presentation are:

R, $t^2 = t$,

$t a_5 t = \cdots = a_5 t a_5 t$,

$t a_i = a_i t \quad (1 \leq i \leq n-2)$.



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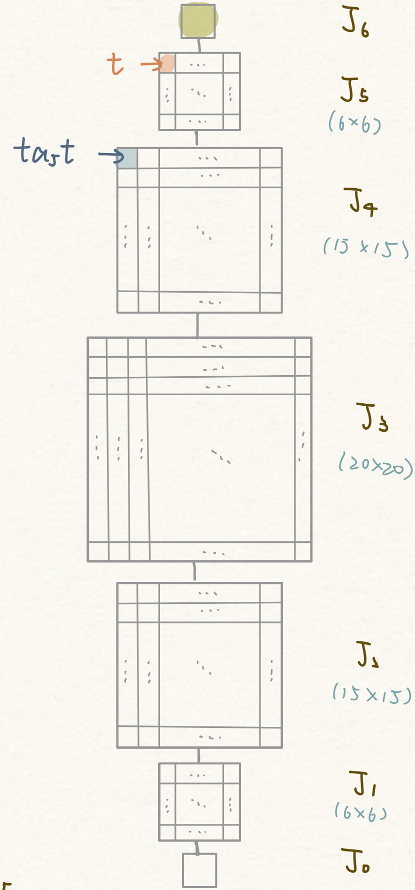

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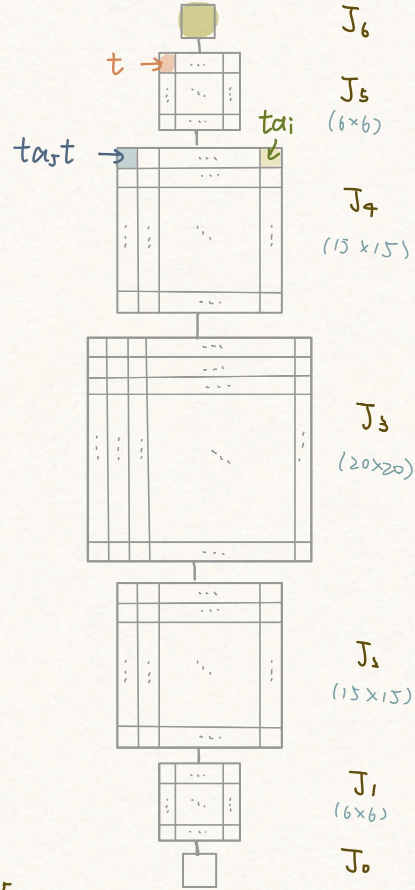

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Presentation for $\mathcal{I}_n \wr \mathcal{S}_n$

Proposition (East, 2006)

For $1 \leq i \leq n$, define λ_i by $x\lambda_i = \begin{cases} x & \text{if } x < i \\ x+1 & \text{if } i \leq x < n \end{cases}$, and ρ_i by

$x\rho_i = \begin{cases} x & \text{if } x < i \\ x-1 & \text{if } i < x \leq n. \end{cases}$ For $1 \leq j \leq n-2$,

define s_i by $x s_i = \begin{cases} x & \text{if } x \neq i, i+1, n \\ i+1 & \text{if } k = i \\ i & \text{if } k = i+1 \end{cases}$.

Define and alphabet $LUSUR$ where the elements in L, S, R are in one-one correspondence with λ_i's, s_i's, ρ_i's respectively. Then $\mathcal{I}_n \wr \mathcal{S}_n$ has presentation

$\langle LUSUR \mid (L1\text{-}L2), (R1\text{-}R2), (RL1\text{-}RL3),$
$(S1\text{-}S4), (SL1\text{-}SL4), (RS1\text{-}RS4) \rangle.$

$\lambda_i\lambda_j = \lambda_{j+1}\lambda_i$	if $1 \leq i \leq j \leq n-1$	(L1)
$\lambda_i\lambda_n = \lambda_i$	if $1 \leq i \leq n$	(L2)
$\rho_j\rho_i = \rho_i\rho_{j+1}$	if $1 \leq i \leq j \leq n-1$	(R1)
$\rho_n\rho_i = \rho_i$	if $1 \leq i \leq n$.	(R2)

$$\rho_i\lambda_j = \begin{cases} \lambda_n\lambda_{j-1}\rho_i & \text{if } 1 \leq i < j \leq n \\ \lambda_n = \rho_n & \text{if } 1 \leq i = j \leq n \\ \lambda_n\lambda_j\rho_{i-1} & \text{if } 1 \leq j < i \leq n. \end{cases} \qquad \text{(RL1–RL3)}$$

$s_i\lambda_n = \lambda_n s_i = s_i$	for all i	(S1)
$s_i^2 = \lambda_n$	for all i	(S2)
$s_i s_j = s_j s_i$	if $\lvert i-j\rvert > 1$	(S3)
$s_i s_j s_i = s_j s_i s_j$	if $\lvert i-j\rvert = 1$.	(S4)

$$s_i\lambda_j = \begin{cases} \lambda_n\lambda_j s_i & \text{if } 1 \leq i < j-1 \leq n-2 \\ \lambda_n\lambda_{j-1} & \text{if } 1 \leq i = j-1 \leq n-2 \\ \lambda_n\lambda_{j+1} & \text{if } 1 \leq i = j \leq n-2 \\ \lambda_n\lambda_j s_{i-1} & \text{if } 1 \leq j < i \leq n-2 \end{cases} \qquad \text{(SL1–SL4)}$$

$$\rho_j s_i = \begin{cases} s_i\rho_j\rho_n & \text{if } 1 \leq i < j-1 \leq n-2 \\ \rho_{j-1}\rho_n & \text{if } 1 \leq i = j-1 \leq n-2 \\ \rho_{j+1}\rho_n & \text{if } 1 \leq i = j \leq n-2 \\ s_{i-1}\rho_j\rho_n & \text{if } 1 \leq j < i \leq n-2. \end{cases} \qquad \text{(RS1–RS4)}$$

East's presentation for $I_5 = I_6 \setminus S_6$

- $s_4 \lambda_4 = \lambda_6 \lambda_5$ is a relation in (SL3), where

$$s_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 4 & - \end{pmatrix},$$

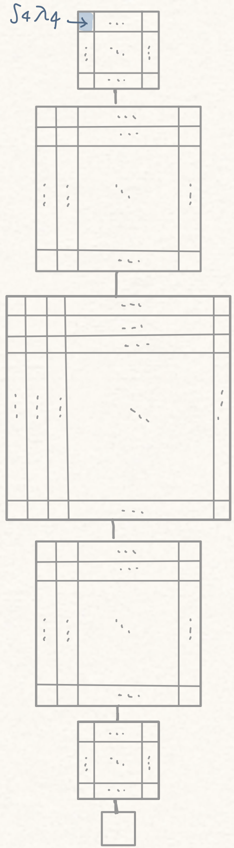
$$\lambda_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 6 & - \end{pmatrix}.$$

- $\rho_4 s_2 = s_3 \rho_4 \rho_n$ is a relation in (RS1), where

$$\rho_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & - & 4 & 5 \end{pmatrix},$$

$$s_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 4 & 5 & - \end{pmatrix}.$$

- $\rho_4^2 = \rho_4 \rho_5$ is a relation in (R1).



J_5 (6×6)

J_4 (15×15)

J_3 (20×20)

J_2 (15×15)

J_1 (6×6)

J_0

East's presentation for $I_5 = I_6 \setminus S_6$

- $s_4\lambda_4 = \lambda_6\lambda_5$ is a relation in (SL3), where

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- $\rho_4 s_2 = s_3\rho_4\rho_n$ is a relation in (RS1), where

$$\rho_4 = \begin{pmatrix}1&2&3&4&5&6\\1&2&3&-&4&5\end{pmatrix},$$

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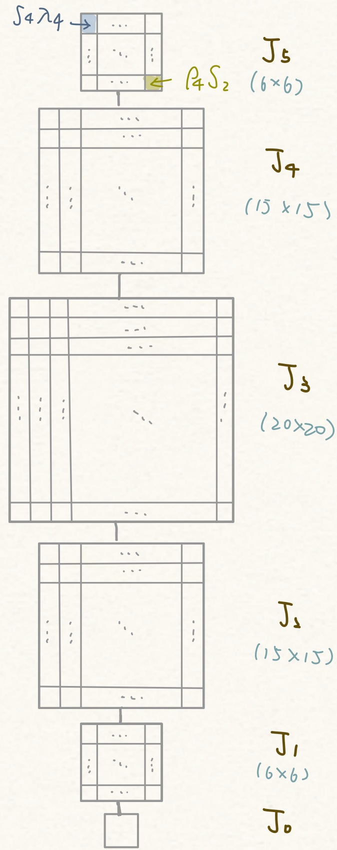
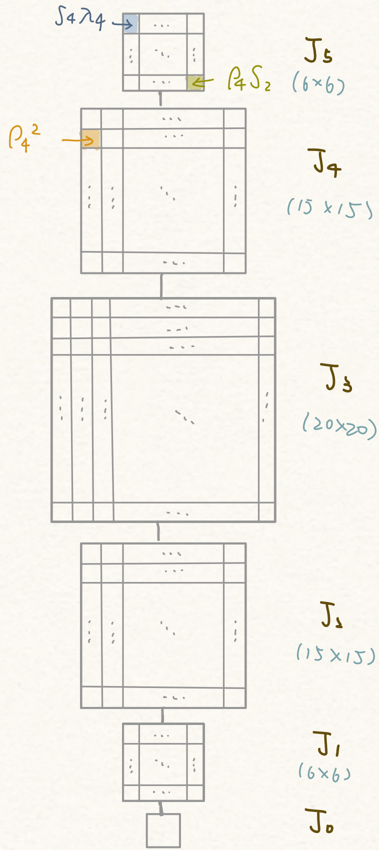


East's presentation for $I_5 = I_6 \setminus S_6$



- $s_4\lambda_4 = \lambda_6\lambda_5$ is a relation in (SL3), where

$$s_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 4 & - \end{pmatrix},$$

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- $\rho_4 s_2 = s_3\rho_4\rho_n$ is a relation in (RS1), where

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- $\rho_4^2 = \rho_4\rho_5$ is a relation in (R1).

Some other presentations for the transformation semigroups and the singular part

- O. Ganyushkin, V. Mazorchuk. Classical Finite Transformation Semigroups (2008) (presentation for PT_n)
- J. East, Defining relations for idempotent generators in finite partial transformation semigroups (2013) ($PT_n \setminus S_n$)
- J. D. Mitchell, M. T. Whyte, Short presentations for transformation monoids (2024) (short presentations for I_n, T_n, PT_n)
- J. East, A symmetrical presentation for the singular part of the symmetric inverse monoid (2015) ($I_n \setminus S_n$)

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Question: How about a general ideal I_k of $\mathcal{T}_n$, $\mathcal{I}_n$, $\mathcal{PT}_n$?

Relational depth

> Consider a finite semigroup S whose $\mathcal{J}$-classes form a chain. Let $\mathcal{P} = \langle A \mid R \rangle$ be a presentation for S. Let $w \in A^+$, and $(u=v) \in R$.

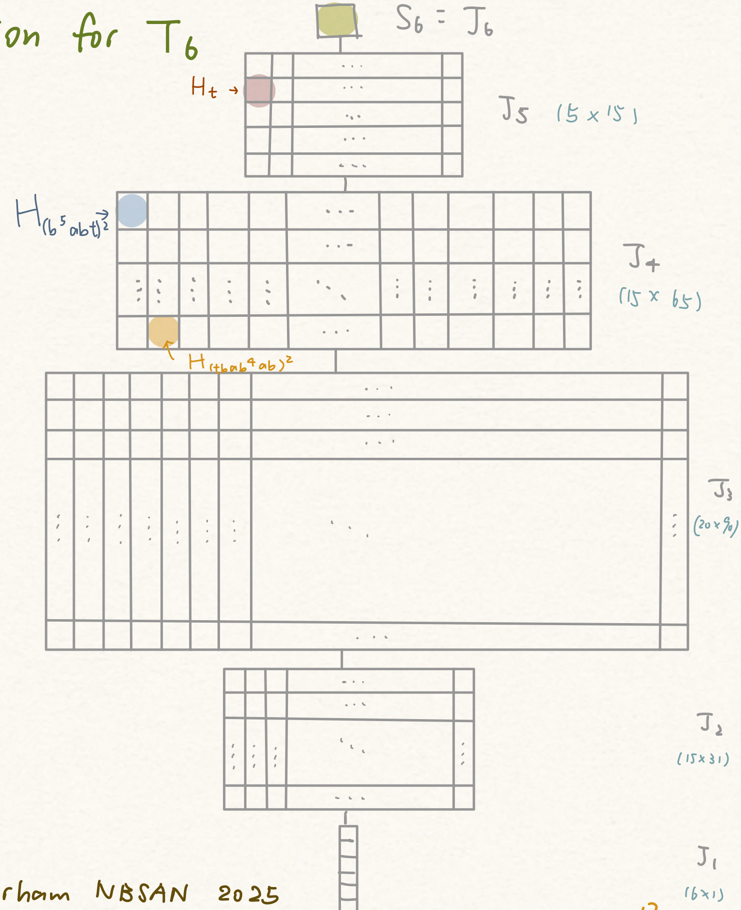
- depth (J) is the minimal length d of a maximal chain of $\mathcal{J}$-classes $J = J_1 < J_2 < \cdots < J_d$ in S.
- depth (S) := depth (J_S)
- depth (w) is the depth of the element it represents in S
- depth $(u=v)$:= depth(u) $\quad$ (= depth(v))
- depth $(\mathcal{P})$ is the maximal depth of a defining relation in $\mathcal{P}$

> The relational depth of S is

$$\text{depth}(S) := \min\{\text{depth}(\mathcal{P}) : \mathcal{P} \text{ is a presentation for } S\}.$$

Aĭzenštat's presentation for T_6

Relations in Aĭzenštat's presentation are in the top 3 J-classes, so has depth 3, and depth$(T_n) \le 3$.



Depth (of the Cayley table presentation)

> Recall: $\operatorname{depth}(S) = \min\{\operatorname{depth}(P) : P$ is a presentation for $S\}$.

> Suppose that $S = J_1 \cup \cdots \cup J_k$ is a semigroup with chain-like $\mathcal{J}$-classes. Let $U = J_i \cup \cdots \cup J_k$. The Cayley table presentation restricted to U is

$$C_i = \langle x_\alpha : \alpha \in J_i \cup \cdots \cup J_k \mid x_\alpha x_\beta = x_{\alpha\beta} : \alpha, \beta, \alpha\beta \in J_i \cup \cdots \cup J_k \rangle.$$

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Theorem

Let S be a semigroup whose $\mathcal{J}$-classes form a chain $J_1 \cup \cdots \cup J_k$. The depth of S is $k-i+1$ where i is the largest number for which C_i defines S.

Relational depth of transformation semigroups & ideals

Theorem

Let $n \geq 3$, let $S \in \{T_n, I_n, PT_n\}$. Define $\varepsilon = \begin{cases} 0 & (S = I_n, PT_n) \\ 1 & (S = T_n) \end{cases}$.

For $m \in [\varepsilon, n]$, let I_m be the ideal of all (partial) maps in S of rank at most m. Then

$$\operatorname{depth}(I_m) = \begin{cases} m - \max(\varepsilon, 2m-n) + 1 & \text{if } m < n \\ 3 & \text{if } m = n. \end{cases}$$

Relational depth of (ideals of) $\mathcal{I}_n$

Theorem

Let $n \geq 3$, and for $m \in [0, n]$, let I_m be the ideal of all partial bijections in $\mathcal{I}_n$ of rank at most m. Then

$$\operatorname{depth}(I_m) = \begin{cases} m - \max(0, 2m-n) + 1 & \text{if } m < n \\ 3 & \text{if } m = n. \end{cases}$$

- In other words, when $m \leq \frac{n}{2}$, C_i defines I_m if and only if $i = 0$.
- If $m > \frac{n}{2}$, C_i defines I_m if and only if $i \leq 2m-n$.

Idea of the proof

let $m<n$. and let $\mathcal{I}_m$ be an ideal of $\mathcal{I}_n$.

- Find a lower bound for the depth:

Lemma.

If $m > r > \max(2m-n, 0)$, then $\mathcal{C}_r = \langle X_{r,m} \mid R_{r,m} \rangle$ does not define $\mathcal{I}_m$.

Idea of the proof

Let $m<n$. and let I_m be an ideal of I_n.

➤ Find a lower bound for the depth:

Lemma.

If $m>r>\max(2m-n,0)$. then $\mathcal{C}_r = \langle X_{r,m} \mid R_{r,m} \rangle$ does not define I_m.

(sketch proof). Suppose $r>\max(2m-n,0)$. Choose $\alpha.\beta.\gamma \in J_m$ such that $\alpha\beta.\ \beta\gamma=\alpha\beta\gamma \in J_r$. Show that the relation $x_\alpha x_\beta x_\gamma = x_\beta x_\gamma$ is not a consequence of $R_{r,m}$.

Multiplication depth

Let $S = J_1 \cup \cdots \cup J_k$ be a semigroup whose $\mathcal{J}$-classes form a chain.

- The multiplication depth of S is defined to be

  $$\text{m. depth}(S) := k - i + 1$$

  where i is the smallest number such that there exist $\alpha, \beta \in J_k$ with $\alpha\beta \in J_i$.

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- Following from the lemma, if there exists $\alpha, \beta, \gamma \in J_m$ s.t. $\alpha\beta, \beta\gamma = \alpha\beta\gamma \in J_r$, then

  $$\text{depth}(I_m) \geq \text{m.depth}(I_m).$$

Idea of the proof (continued)

Let $m<n$, and let I_m be an ideal of I_n.

- Find an upper bound for the depth:

Lemma.

Let $r = 2m-n$. Then $\mathcal{C}_r = \langle X_{r,m} \mid R_{r,m} \rangle$ defines a presentation for I_m.

(sketch proof). We show that some important relations can be deduced from the relations in $R_{r,m}$.

Idea of the proof (continued)

Show that relations of the form:

i). $x_\alpha x_\beta = x_\gamma x_\delta$ $(\alpha, \beta, \gamma, \delta \in J_r,\ \alpha\beta = \gamma\delta \in J_{r-1})$

ii). $x_\alpha x_\beta = x_\gamma x_\delta$ $(\mathrm{rank}\,\gamma, \mathrm{rank}\,\delta > r,$
$\alpha, \beta \in J_r,\ \alpha\beta = \gamma\delta \in J_{r-1})$

iii). $x_\alpha x_\beta x_\gamma = x_{\alpha'} x_{\gamma'}$ $(\alpha, \beta, \gamma, \alpha', \gamma' \in J_r,$
$\alpha\beta\gamma = \alpha'\gamma' \in J_{r-1})$

are consequences of $R_{r,m}$.

Idea of the proof (continued)

Show that relations of the form:

i). $x_\alpha x_\beta = x_\gamma x_\delta$ ($\alpha, \beta, \gamma, \delta \in J_r$, $\alpha\beta = \gamma\delta \in J_{r-1}$)

ii). $x_\alpha x_\beta = x_\gamma x_\delta$ (rank γ, rank $\delta > r$, $\alpha, \beta \in J_r$, $\alpha\beta = \gamma\delta \in J_{r-1}$)

'replace subwords of the same length'

iii). $x_\alpha x_\beta x_\gamma = x_{\alpha'} x_{\gamma'}$ ($\alpha, \beta, \gamma, \alpha', \gamma' \in J_r$, $\alpha\beta\gamma = \alpha'\gamma' \in J_{r-1}$)

'change the length'

are consequences of $R_{r,m}$.

▶ $w = x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} \cdots x_{\alpha_k}$ $\longrightarrow$ $w' = x_{\beta_1} x_{\beta_2}$

($\alpha_i \in J_r \cup J_{r+1} \cup \cdots \cup J_m$) ($\beta_1, \beta_2 \in J_r$)

Example: Relational depth of $\mathcal{I}_9$ and its ideals

	relational depth	c_i where i is the largest
I_0	1	c_0
I_1	2	c_0
I_2	3	c_0
I_3	4	c_0
I_4	5	c_0
I_5	5	c_1
I_6	4	c_3
I_7	3	c_5
I_8	2	c_7
$\mathcal{I}_9$	3	c_7

Relational & multiplication depth

- For the ideals of T_n, I_n, and PT_n,

  relational depth = multiplication depth.

  - Is this always true?

Relational depth = multiplication depth ? NO.

Example. Let I_m be a proper ideal of T_n, where $m > \frac{n+1}{2}$,

and $\mathrm{depth}(I_m) = m-r+1$ where $r = 2m-n$.

Define the Rees quotient semigroup

$$I_m' = I_m \setminus I_{r-1}.$$

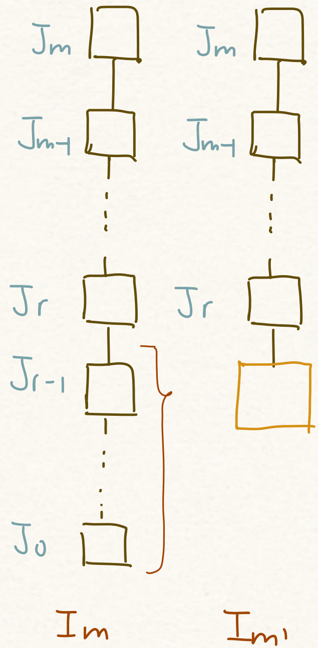
- multiplication depth is the same

  $$\mathrm{m.depth}(I_m') = m-r+1.$$

- $\langle X_{r,m} \mid R_{r,m} \rangle$ does not define I_m'

In particular,

$$\mathrm{depth}(I_m') = m-r+2 = \mathrm{m.depth}(I_m')+1.$$



Other interesting semigroups

- the semigroup of order preserving maps (✓)

- diagram monoids – Partition monoid, 

  Brauer monoid (✓). Temperley-Lieb monoid

  odd & even cases

- semigroup of $n \times n$ matrices

⋮

Thank you =)