Pretzel Monoids

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Left Adequate Monoids

Definition

A monoid M is *left adequate* if:

1. Idempotents of M commute;
2. For all $a \in M$, there exists a unique idempotent $a^+ \in E(M)$ such that

$$\forall x, y \in M \quad xa = ya \iff xa^+ = ya^+.$$

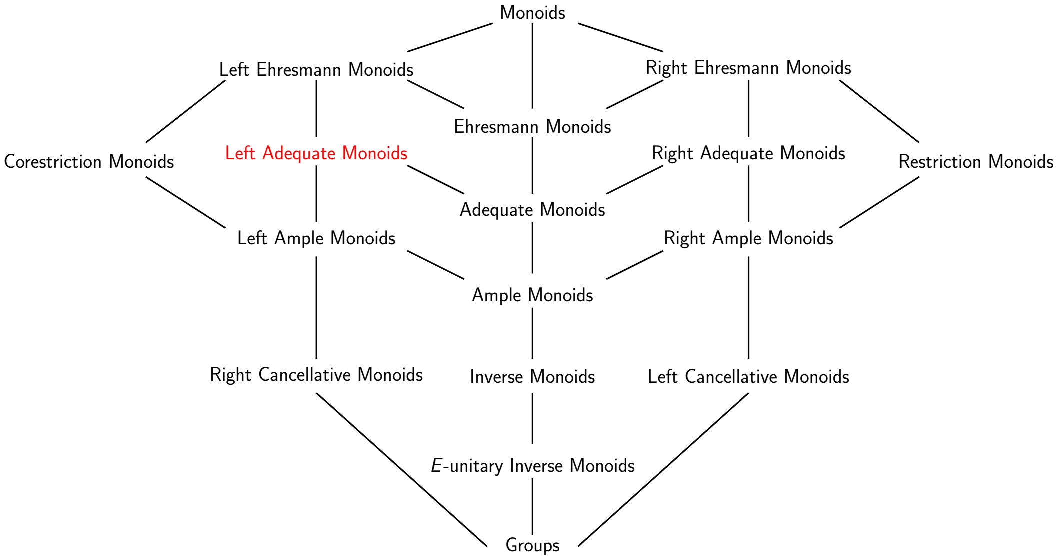
Definition

Equivalently, a monoid equipped with a unary operation $+$ is called *left adequate* if it satisfies the defining identities:

$$a^+a = a, \quad (a^+b^+)^+ = a^+b^+, \quad a^+b^+ = b^+a^+, \quad (ab)^+ = (ab^+)^+,$$

$$a^2 = a \to a = a^+ \quad \text{and} \quad ac = bc \to ac^+ = bc^+.$$

A Big Diagram



A Potted History

- Kilp 1973, *Commutative monoids all of whose principal ideals are projective*.
- Fountain 1976, *Right PP monoids with central idempotents*.

Remark. A monoid M is *right PP* (modernly known as *right abundant*) if and only if every $\mathcal{L}^*$-class contains an idempotent. Very reminiscent of regular and inverse semigroups!

- Fountain 1977, *A class of right PP monoids*.
- Fountain 1979, *Adequate semigroups*.

Definition

A right adequate semigroup is called *right ample* if $eM \cap aM = eaM$ for all $e \in E(M)$, $a \in M$.

Theorem (Fountain 1977)

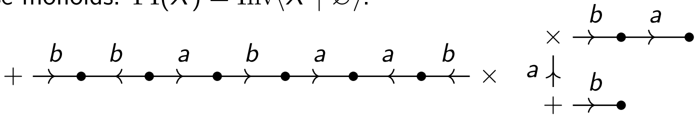
Let M be a right ample monoid. Then S is the image of some "proper" right ample monoid under some morphism θ where $a\theta = b\theta \implies a\mathcal{L}^* b$.

- See also Hollings 2009, *From right PP monoids to restriction semigroups: a survey*.

A Potted History 2

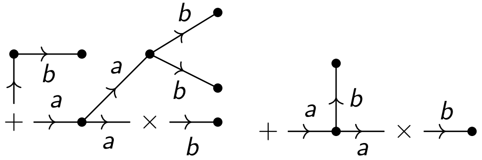
- Munn 1974, Free inverse monoids. $\mathrm{FI}(X) = \mathrm{Inv}\langle X \mid \varnothing \rangle$.

$$bb^{-1}abaa^{-1}b^{-1}$$



- Fountain 1991, *Free right type A semigroups*.
- Fountain, Gomes, Gould 2009, *The free ample monoid.* $\mathrm{FLAm}(X) = \mathrm{LAm}\langle X \mid \varnothing \rangle$.

$$(ab)^{+}a(ab^{+}b)^{+}ab^{+}$$



- Kambites 2009-11, *Free adequate semigroups* $\mathrm{FLAd}(X) = \mathrm{LAd}\langle X \mid \varnothing \rangle$.

Free Left Adequate Monoids

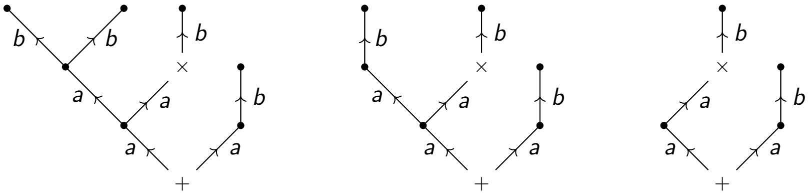
For trees in $\mathrm{FLAd}(X)$, first draw the tree as in $\mathrm{FLAm}(X)$...

Definition

The *retraction* of a tree is the smallest image under an idempotent endomorphism which fixes $+$ and $\times$.

Fact: The retraction of a tree is unique up to isomorphism.

$(ab)^+a(ab^+b)^+ab^+$



The Goal

Question: What results in inverse or cancellative semigroups can we generalise to (left) adequate or ample monoids?

- McAlister's covering theorem and the P-theorem (Fountain 1970s).
- Munn trees and free inverse monoids (Fountain, Gomes, Gould, Kambites, 2000s).
- Stephen's procedure...? (Likely very hard!)
- *Expansions*!

Semigroup Expansions

Definition (Birget, Rhodes 1984)

Let $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathbf{Sgp}$. An *expansion of* $\mathcal{C}$ *to* $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that there is a natural transformation $\eta : F \implies \iota_{C \to D}$ whose components η_S are all surjective.

I.e. for all $S \in \mathcal{C}$, there is a semigroup $F(S) \in \mathcal{D}$ and a surjective morphism $\eta_S : F(S) \to S$ such that whenever $\tau : S \to T$ is a morphism, there is a morphism $F(\tau) : F(S) \to F(T)$ making the following diagram commute:

$$\begin{array}{ccc} F(S) & \xrightarrow{F(\tau)} & F(T) \\ \eta_S \big\downarrow & & \big\downarrow \eta_T \\ S & \xrightarrow{\quad \tau \quad} & T \end{array}$$

Theorem (Birget, Rhodes 1984 / Szendrei 1989)

There is an expansion $\mathrm{Sz} : \mathbf{Gp} \to \mathbf{FInv}$ *given by* $\mathrm{Sz}(G) = \{(H, g) : H \subseteq G$ *finite and* $1, g \in H\}$.

Theorem (Szendrei 1989)

Sz *is left adjoint to the maximal group image functor* $\sigma^\natural : \mathbf{FInv} \to \mathbf{Gp}$.

Expansions of other Categories

Recall that $\mathrm{FI}(X)$ was constructed by 'tracing' the Cayley graph of $\mathrm{FG}(X)$... what about other Cayley graphs?

Theorem (Margolis, Meakin 1989)

*Let X be a set and let G be an X-generated group. There is an expansion $\mathcal{M}$: **XGp** $\to$ **XEInv** given by $\mathcal{M}(G) = \{(\Gamma, g) : \Gamma$ is a finite connected subgraph of $\mathrm{Cay}(G), 1, g \in V(\Gamma)\}$. $\mathcal{M}$ is left adjoint to the maximal group image functor $\sigma^\natural$: **XEInv** $\to$ **XGp**.*

Theorem (Gould 1996, + Gomes 2000)

Let X be a set and let M be an X-generated monoid. Define

$$\mathcal{G}(M) = \{(\Gamma, m) : \Gamma \text{ is a finite connected subgraph of } \mathrm{Cay}(M), 1, m \in V(\Gamma)\}.$$

*Then $\mathcal{G}$ forms expansions **XRC** $\to$ **XPLAm** and **XU** $\to$ **XPWLAm**. Moreover, $\mathcal{G}$ is left adjoint to taking the maximal right cancellative image and maximal unipotent image respectively.*

Question

Can we find an expansion **XRC** $\to$ **XLAd**? Preferably with some graphical interpretation?

Pretzels!

Fix a set X and an X-generated right cancellative monoid C.

Definition

An *idempath* in an X-labelled digraph Γ is a path labelled by a word $x_1 x_2 \cdots x_n$ which is equal to the identity in C. We take the empty path with label ϵ to have $\epsilon =_C 1$.
An *idempath identification* in Γ is the process of 'cycling up' an idempath.

Lemma (H., Kambites, Szakács 2024)

Given a tree $T \in \mathrm{FLAd}(X)$, there exists a unique graph obtainable by sequentially performing all non-trivial idempath identifications (in any order) to T.

Definition

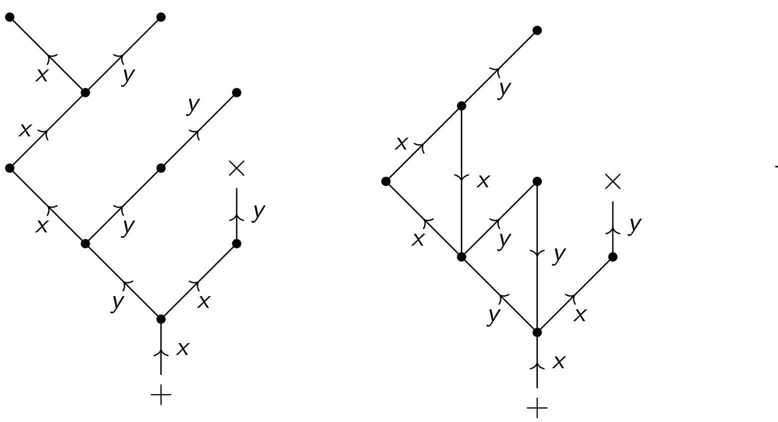
Given any tree $T \in \mathrm{FLAd}(X)$, perform the following:

1. Idempath identify as far as possible...
2. ...then retract anything in the result which can retract (take minimal image under idempotent graph endomorphisms).

We call the (uniquely obtained) result the *pretzel* of T, denoted $\overline{\widetilde{T}}$.

Example

Take $X = \{x, y\}$ and $C = C_3 \times C_3 = \mathrm{Mon}\langle x, y\rangle$.



T $\overline{\widetilde{T}}$

$(2, 1, 0)$-algebras

Define a multiplication $\overline{\widetilde{S}} \cdot \overline{\widetilde{T}}$ on pretzels as follows:

1. Glue $\overline{\widetilde{T}}$ to $\overline{\widetilde{S}}$, start-to-end.
2. Pretzel-ify the result (note that new idempaths could have been created!).

Define a unary operation $+$ on pretzels as follows:

1. Relabel the end vertex of $\overline{\widetilde{T}}$ to be the start vertex.
2. Pretzel-ify the result (note that new retractions might be possible!).

Theorem (H., Kambites, Szakács 2024)

The set of all pretzels $\mathcal{PT}(C; X)$ forms an X-generated left adequate monoid.

Theorem (H., Kambites, Szakács 2024)

$\mathcal{PT}(C; X) \cong \mathbf{LAd}\langle X \mid w^2 = w \text{ for } w \in X^* \text{ s.t. } w =_C 1\rangle.$

Margolis-Meakin Expansions vs. Pretzels

Properties of $\mathcal{M}(G)$

1. $\mathcal{M}(\mathrm{FG}(X)) \cong \mathrm{FI}(X)$.
2. $\mathcal{M}(G)$ is finite $\iff$ G is finite.
3. Elements are subgraphs of $\mathrm{Cay}(G)$.
4. $\mathcal{M}(G) \cong \textbf{Inv}\langle X \mid w^2 = w \text{ for } w \in X^* \text{ s.t. } w =_G 1\rangle$.
5. $\mathcal{M}$ defines an expansion **XGp** $\to$ **XEInv**.

Properties of $\mathcal{PT}(C)$

1. $\mathcal{PT}(X^*) \cong \mathrm{FLAd}(X)$.
2. $\mathcal{PT}(C)$ is finite $\iff$ C is finite $\implies$ C is a group.
3. Elements are trees of strongly connected subgraphs of $\mathrm{Cay}(C)$.
4. $\mathcal{PT}(C; X) \cong \textbf{LAd}\langle X \mid w^2 = w \text{ for } w \in X^* \text{ s.t. } w =_C 1\rangle$.

Theorem (H., Kambites, Szakács 2024)

$\mathcal{PT}$ defines an expansion **XRC** $\to$ **XLAd**.

Open Questions and What's Next

- What about right adequate and two-sided adequate pretzel monoids?
- Can we find geometric interpretations of other analogues of Margolis-Meakin expansions in the left adequate setting, perhaps one such that $M(C)$ has maximal right cancellative image C?

Proposition

The maximal right cancellative image of $\mathcal{PT}(C)$ is $\mathbf{RC}\langle X \mid w = 1 \text{ for } w \in X^* \text{ s.t. } w =_C 1\rangle$. In particular, it is two-sided cancellative.

Proposition

$\mathcal{PT}$ is not left adjoint to the maximal right cancellative image functor.

- Can we apply similar pretzel-style techniques in F-inverse land? In particular for the free F-inverse monoid...?
- What about other interesting presentations of (left) adequate monoids?

Thank you!