Topologies on the Symmetric Inverse Monoid

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- A *semigroup topology* for a semigroup $(S, \cdot)$ is any topology on S under which the multiplication map

$$(a, b) \mapsto a \cdot b$$

  is continuous.
- An *inverse semigroup topology* for an inverse semigroup $(I, \cdot)$ is any topology on I under which the maps

$$(a, b) \mapsto a \cdot b \quad \text{and} \quad a \mapsto a^{-1}$$

  are continuous.
- An inverse semigroup topology on a group is called a *group topology*.

Why Topological Algebra?

Some good ways of using Topology in Semigroup Theory:

1. Fix a semigroup S. What kind of topologies does S admit?
2. Conversely, fix some topological properties (say compact & Hausdorff). What can you say about semigroups admitting such topologies?
3. Fix a semigroup S and a semigroup topology τ for S. Study topologically-algebraic problems:
   - What are the subsemigroups of S which are closed (or open, compact, ...) under τ?
   - What is the least number of elements of S that generates a subsemigroup of S which is dense under τ? ("topologically generating S")

For point 3 to be interesting and meaningful, we need to agree on a τ which is (i) 'natural' for S and (ii) 'nice' in a topological sense.

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There are nine, increasingly stronger, “separation axioms”

$$T_0 \Leftarrow T_1 \Leftarrow T_2 \Leftarrow T_{2\frac{1}{2}} \Leftarrow T_3 \Leftarrow T_{3\frac{1}{2}} \Leftarrow T_4 \Leftarrow T_5 \Leftarrow T_6.$$

They describe ways in which points in the space may be separated by open sets. For example, a topological space S is...

- ... T_1 (Fréchet) if for all distinct $x, y \in S$, there exists an open neighbourhood of x which does not contain y;
- ... T_2 (Hausdorff) if all distinct $x, y \in S$ have disjoint open neighbourhoods.

$U \subseteq S$ is a *neighbourhood* of $x \in S$ if $x \in V \subseteq U$ for some open $V \subseteq S$.

Properties that make topologists happy

Part 2. Having **not too many** open sets

A topological space S is. . .

- . . . *separable* if S has a countable dense subset;
- . . . *compact* if every open cover of S may be reduced to a finite subcover;
- . . . *connected* if no open set (other than $\emptyset$ and S) is also closed;

A topological space (S, τ) is...

- ... *second-countable* if τ has a countable basis;
- ... *metrizable* if τ is induced by a metric on S;
- ... *completely metrizable* if τ is induced by a complete metric on S;
- ... *Polish* if S is completely metrizable and separable.
- ... *locally compact* if every point in S has a compact neighbourhood.

Natural topologies for a semigroup

$(S, \cdot)$ is a semigroup. What semigroup topology should we give S?
Two approaches:

1. **From context:** What kind of object is S? Does the set S already come with a topology we care about? Examples:
   - the real numbers under addition $(\mathbb{R}, +)$
   - general linear groups $GL_n(\mathbb{R})$
2. **Purely algebraic:** Ignore the context (if any) of the set S as an object and consider topologies that may be defined on any abstract semigroup $(S, \cdot)$. Examples:
   - Minimal topologies which are T_1, Hausdorff, ...
   - Maximal topologies which are compact, second-countable,...
   - Topologies defined via algebraic equations (Zariski topologies).

We will now consider some of these "purely algebraic" topologies in more detail.

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Some minimal topologies on any semigroup S

Let S be a semigroup.

- *semigroup Fréchet Markov topology* on $S :=$ intersection of all all T_1 semigroup topologies on S.
- *semigroup Hausdorff Markov topology* on $S :=$ intersection of all Hausdorff semigroup topologies on S.

Warning: Hausdorffness may be lost

The semigroup Markov topologies are both T_1 but neither is necessarily T_2.

Warning: joint continuity may be lost

The semigroup Markov topologies may not be semigroup topologies! They may only be "shift continuous".

We analogously define inverse Markov topologies on a group or inverse semigroup G and the same warnings apply.

Semigroup topologies vs shift continuous topologies

- Recall: under a semigroup topology on $(S, \cdot)$ the multiplication map
$$(a, b) \mapsto a \cdot b$$
is continuous. (Multiplication is a function from $S \times S$ to S.)
- Under a *shift continuous* topology on S, for every fixed $s \in S$, the maps given by left or right "shifts by s"
$$a \mapsto s \cdot a \text{ and } a \mapsto a \cdot s$$
are continuous. (The shift maps are functions from S to S.)
- semigroup topology $\implies$ shift continuous topology (but the converse is false)

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Semigroup polynomials

A *semigroup polynomial* over a semigroup $(S, \cdot)$ is a function $P : S \to S$ of the form

$$(x)P = a_0 \cdot x \cdot a_1 \cdot x \cdot \ldots a_{n-1} \cdot x \cdot a_n \qquad \text{for all } x \in S$$

where $a_0, a_1, \ldots, a_n \in S^1$.

Inverse semigroup (including group) polynomials

An *inverse semigroup polynomial* on an inverse semigroup G is of the form

$$(x)P = a_0 \cdot x^{\epsilon_1} \cdot a_1 \cdot x^{\epsilon_2} \cdot \ldots a_{n-1} \cdot x^{\epsilon_n} \cdot a_n \qquad \text{for all } x \in S$$

where $a_0, a_1, \ldots, a_n \in G^1$ and $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$

The semigroup Zariski topology on a semigroup S

The topology generated by the sets of the form

$$\{x \in S : (x)P \neq (x)Q\}$$

over all semigroup polynomials P and Q over S.

The inverse Zariski topology on an inverse semigroup G

The topology generated by the sets of the form

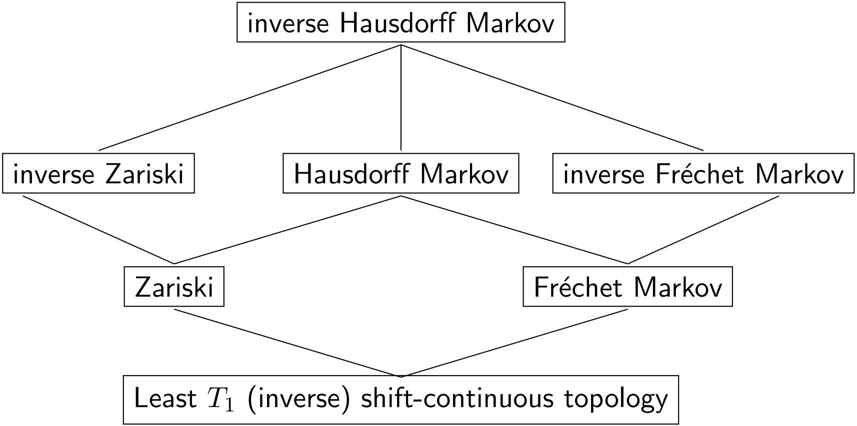
$$\{x \in G : (x)P \neq (x)Q\}$$

over all inverse semigroup polynomials P and Q over G.

The Zariski topology is always T_1 and shift-continuous and is contained in every Hausdorff semigroup topology on S.

Properties of minimal topologies

The containment of the “minimal” topologies:



A maximal topology: automatic continuity

Recall: a topology is second-countable if it can be given by a countable basis (or sub-basis).

Definition

Let τ_{AC} be the **union** of all second-countable semigroup topologies on S.

Properties of τ_{AC}:

- τ_{AC} is always a semigroup topology for S.
- If τ_{AC} is itself second-countable, then τ_{AC} is the maximal second-countable semigroup topology on S.
- S has *automatic continuity* under τ_{AC}: every homomorphism from S to any second-countable topological semigroup is continuous.

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Today's semigroups of interest (at least in this talk)

We will consider topologies on the following semigroups acting via (partial) functions on a set X.

- The *full transformation semigroup* X^X consisting of all functions $f : X \to X$;
- The *symmetric group* $\mathrm{Sym}(X)$ consisting of a bijections $f : X \to X$;
- The *symmetric inverse monoid* I_X consisting of all bijections between subsets of X.

The operation in each case is composition of (partial) functions. For simplicity, we will only consider the case when X is countably infinite. So we let $X = \mathbb{N} = \{0, 1, 2, 3 \ldots\}$.

A topology for $\mathbb{N}^{\mathbb{N}}$

Is there context? Does the set $\mathbb{N}^{\mathbb{N}}$ already already have a natural topology? Yes!

- Note that $\mathbb{N}^{\mathbb{N}}$ is the infinite Cartesian product $\mathbb{N}^{\omega} = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots$, i.e. all sequences over $\mathbb{N}$. (Think of $f : \mathbb{N} \to \mathbb{N}$ as $(f(0), f(1), f(2), \ldots)$.)
- The natural (from a topologist's point of view) topology on a Cartesian product of topological spaces is the (Tychonoff) product topology.
- If we give each copy of $\mathbb{N}$ the discrete topology (which seems natural), then $\mathbb{N}^{\omega}$ is the so-called *Baire space*.
- A sub-basis for the product topology is given by the sets $\{f \in \mathbb{N}^{\mathbb{N}} : (m)f = n\}$ over all $m, n \in \mathbb{N}$.
- This topology for $\mathbb{N}^{\mathbb{N}}$ is called the *pointwise topology*.

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The pointwise topology is nice

The Baire space ($\mathbb{N}^{\mathbb{N}}$ under the pointwise topology) has the following properties:

- $\mathbb{N}^{\mathbb{N}}$ is Polish (completely metrizable and separable).
- In particular, $\mathbb{N}^{\mathbb{N}}$ satisfies all separation axioms $T_0, \ldots, T_6$ and is second-countable.
- $\mathbb{N}^{\mathbb{N}}$ is far from being (even locally) compact: it contains no compact neighbourhoods.
- $\mathbb{N}^{\mathbb{N}}$ is *totally disconnected*: the only connected subspaces are single points.
- Every Polish space is the continuous image of $\mathbb{N}^{\mathbb{N}}$.
- The subspace $\mathrm{Sym}(\mathbb{N})$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

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The pointwise topology is natural for $\mathbb{N}^\mathbb{N}$ and $\mathrm{Sym}(\mathbb{N})$

Under the pointwise topology:

- $\mathbb{N}^\mathbb{N}$ is a topological semigroup;
- Sym$(\mathbb{N})$ is a topological group;
- A submonoid of $\mathbb{N}^\mathbb{N}$ is closed if and only if it is the endomorphism monoid of a relational structure on $\mathbb{N}$.
- A subgroup of Sym$(\mathbb{N})$ is closed if and only if it is the automorphism group of a relational structure on $\mathbb{N}$.

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The pointwise topology for $\mathrm{Sym}(\mathbb{N})$ as an abstract group

Theorem (Gaughan 1967)

The pointwise topology is the least Hausdorff group topology on $\mathrm{Sym}(\mathbb{N})$*.*

Theorem (Kechris, Rosendal 2004)

The pointwise topology is the unique non-trivial separable group topology on $\mathrm{Sym}(\mathbb{N})$*.*

Corollary

- *the pointwise topology is the unique Polish group topology on Sym(*$\mathbb{N}$*).*
- *the pointwise topology is the inverse Hausdorff Markov, inverse Fréchet Markov, and Zariski topology on* $\mathrm{Sym}(\mathbb{N})$*.*
- *Every homomorphism from* $\mathrm{Sym}(\mathbb{N})$ *into any second-countable topological group is continuous (automatic continuity).*

The pointwise topology on $\mathbb{N}^{\mathbb{N}}$ as an abstract semigroup

Theorem (L. Elliott, J. Jonušas, J.D. Mitchell, Z. Mesyan, M. Morayne, YP 2023)

The pointwise topology is

1. *the least T_1 and shift continuous topology on $\mathbb{N}^{\mathbb{N}}$.*
2. *the maximal second-countable semigroup topology on $\mathbb{N}^{\mathbb{N}}$.*

Corollary

- *The pointwise topology on $\mathbb{N}^{\mathbb{N}}$ is*
  1. *the unique T_1 and second-countable semigroup topology;*
  2. *the unique Polish semigroup topology;*
  3. *the Fréchet Markov, Hausdorff Markov, and Zariski topology.*
- *If S is a second-countable topological semigroup, then every homomorphism $\phi : \mathbb{N}^{\mathbb{N}} \to S$ is continuous.*
- *No T_1 and shift-continuous topology on $\mathbb{N}^{\mathbb{N}}$ is connected or (locally) compact.*

Proof Sketches Part 1

Claim 1: The pointwise topology is the least T_1 and shift continuous topology on $\mathbb{N}^\mathbb{N}$.

Proof:

- Let τ be a shift continuous and T_1 topology for $\mathbb{N}^\mathbb{N}$ and $m, n \in \mathbb{N}$. We need to show that the sub-basic open set $\{f \in \mathbb{N}^\mathbb{N} : (m)f = n\}$ of the pointwise topology is open in τ.
- Let $c_m \in \mathbb{N}^\mathbb{N}$ be constant with image m, $k \neq n$, and $h \in \mathbb{N}^\mathbb{N}$ satisfy $(m)h = n$ and $(x)h = k$ for $x \neq m$.
- Since τ is T_1, $\{c_k\}$ is closed.
- So $\{f \in \mathbb{N}^\mathbb{N} : c_m f h = c_k\}$ is closed since τ is shift-continuous.
- But $\{f \in \mathbb{N}^\mathbb{N} : c_m f h = c_k\} = \{f \in \mathbb{N}^\mathbb{N} : (m)f \neq n\}$.
- So $\{f \in \mathbb{N}^\mathbb{N} : (m)f = n\}$ is open.

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Proof Sketches Part 2

Claim 2: The pointwise topology is the maximal second-countable semigroup topology on $\mathbb{N}^{\mathbb{N}}$.

Property X

A topological semigroup S has *property* **X** with respect to $A \subseteq S$ if: for every $s \in S$ there exists $f_s, g_s \in S$ and $t_s \in A$ such that $s = f_s t_s g_s$ and for every neighbourhood B of t_s the set $f_s(B \cap A)g_s$ is a neighbourhood of s.

Proof (Very Sketchy):

- Show that $\mathbb{N}^{\mathbb{N}}$ has "property **X**" with respect to $\mathrm{Sym}(\mathbb{N})$.
- Conclude that, since the pointwise topology is Polish and the maximal second-countable group topology on $\mathrm{Sym}(\mathbb{N})$, it is the maximal second-countable semigroup topology on $\mathbb{N}^{\mathbb{N}}$.

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Finding a topology on $I_{\mathbb{N}}$: Extending from $\mathrm{Sym}(\mathbb{N})$

What is the right topology on $I_{\mathbb{N}}$?

- No obvious (to me) topology on $I_{\mathbb{N}}$ as a set.
- Try extending the pointwise topology from $\mathrm{Sym}(\mathbb{N})$ to $I_{\mathbb{N}}$?
- Recall: The pointwise topology on $\mathrm{Sym}(\mathbb{N})$ has sub-basic sets $\{f \in \mathrm{Sym}(\mathbb{N}) : (m)f = n\}$ over all $m, n \in \mathbb{N}$.

Topology I_0 on $I_{\mathbb{N}}$

The topology with sub-basic sets $\{f \in I_{\mathbb{N}} : (m, n) \in f\}$ over all $m, n \in \mathbb{N}$.

The good: I_0 is an inverse semigroup topology for $I_{\mathbb{N}}$ and induces the pointwise topology on $\mathrm{Sym}(\mathbb{N})$.
The bad: I_0 is not T_1. (If $f \subseteq g$, then every open neighbourhood of f contains g.)

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Trying for a T_1 topology

Can we find the least T_1 shift-continuous topology for $I_{\mathbb{N}}$? (in the case of $\mathbb{N}^{\mathbb{N}}$, this was the pointwise topology).

- Suppose that τ is a shift-continuous and T_1 topology for $I_{\mathbb{N}}$ and let $m, n \in \mathbb{N}$. For any $x, y \in \mathbb{N}$, let $s_{x,y} = \{(x, y)\} \in I_{\mathbb{N}}$.
- Then

$$\begin{aligned} \{f \in I_{\mathbb{N}} : s_{m,m} f s_{n,n} = s_{m,n}\} &= \{f \in I_{\mathbb{N}} : (m, n) \in f\} \\ \{f \in I_{\mathbb{N}} : s_{m,m} f s_{n,n} = \emptyset\} &= \{f \in I_{\mathbb{N}} : (m, n) \notin f\} \end{aligned}$$

  are both closed.
- So $\{f \in I_{\mathbb{N}} : (m, n) \in f\}$ and $\{f \in I_{\mathbb{N}} : (m, n) \notin f\}$ are open.

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Properties of I_1

Topology I_1 on $I_{\mathbb{N}}$

The topology with the sub-basic sets
$U_{m,n} = \{f \in I_{\mathbb{N}} : (m,n) \in f\}$ and $V_{m,n} = \{f \in I_{\mathbb{N}} : (m,n) \notin f\}$.

Theorem (L. Elliott, J. Jonušas, J.D. Mitchell, Z. Mesyan, M. Morayne, YP 2023)

The topology I_1 on $I_{\mathbb{N}}$ is

1. *Polish and compact(!?);*
2. *the least T_1 and shift continuous topology;*
3. **not a semigroup topology** *but inversion is continuous.*

Can we find a T_1 (or higher) semigroup topology for $I_{\mathbb{N}}$?

Inheriting from $\mathbb{N}^{\mathbb{N}}$

Since $I_{\mathbb{N}}$ embeds in a full transformation semigroup, we can try to inherit a semigroup topology from the pointwise topology:

- Let $\mathbb{N}' = \mathbb{N} \cup \{\diamondsuit\}$ where $\diamondsuit$ represents "undefined".
- For $f \in I_{\mathbb{N}}$ define $f' \in \mathbb{N}'^{\mathbb{N}'}$ by

$$(x)f' = \begin{cases} (x)f & \text{if } x \in \mathrm{dom}(f) \\ \diamondsuit & \text{otherwise} \end{cases}$$

  Then the map $f \mapsto f'$ embeds $I_{\mathbb{N}}$ in $\mathbb{N}'^{\mathbb{N}'}$.
- The pointwise topology on $\mathbb{N}'^{\mathbb{N}'}$ induces a semigroup topology I_2 on $I_{\mathbb{N}}$ via this embedding.

Topology I_2 on $I_{\mathbb{N}}$

The topology with sub-basic sets
$U_{m,n} = \{f \in I_{\mathbb{N}} : (m,n) \in f\}$ and $W_m = \{f \in I_{\mathbb{N}} : m \notin \mathrm{dom}(f)\}$.

Properties of I_2

Topology I_2 on $I_\mathbb{N}$

The topology with sub-basic sets
$U_{m,n} = \{f \in I_\mathbb{N} : (m,n) \in f\}$ and $W_m = \{f \in I_\mathbb{N} : m \notin \mathrm{dom}(f)\}$.

By construction of I_2, we automatically get:

- I_2 is a semigroup topology for $I_\mathbb{N}$;
- I_2 is Polish (since $I_\mathbb{N}$ is closed in $\mathbb{N}'^{\mathbb{N}'}$).

But inversion is not continuous! Embedding $I_\mathbb{N}$ into $\mathbb{N}'^{\mathbb{N}'}$ has broken symmetry.
I_2 has a dual $I_3 = I_2^{-1} = \{U^{-1} : U \in I_2\}$ where
$U^{-1} = \{f^{-1} : f \in U\}$.

Topology I_3 on $I_\mathbb{N}$

The topology with sub-basic sets
$U_{m,n} = \{f \in I_\mathbb{N} : (m,n) \in f\}$ and $W_m = \{f \in I_\mathbb{N} : m \notin \mathrm{im}(f)\}$.

Properties of I_2 and I_3

Topologies I_2 and I_3 on $I_{\mathbb{N}}$

I_2 is the topology with sub-basic sets
$U_{m,n} = \{f \in I_{\mathbb{N}} : (m,n) \in f\}$ and $W_m = \{f \in I_{\mathbb{N}} : m \notin \operatorname{dom}(f)\}$.
I_3 is the topology with sub-basic sets
$U_{m,n} = \{f \in I_{\mathbb{N}} : (m,n) \in f\}$ and $W_m^{-1} = \{f \in I_{\mathbb{N}} : m \notin \operatorname{im}(f)\}$.

Theorem (L. Elliott, J. Jonušas, J.D. Mitchell, Z. Mesyan, M. Morayne, YP 2023)

- *I_2 and I_3 are Polish semigroup topologies for $I_{\mathbb{N}}$;*
- *every T_1 semigroup topology for $I_{\mathbb{N}}$ contains I_2 or I_3;*
- *$I_1 \subsetneq I_2 \cap I_3$ and $I_2 \cap I_3$ is the semigroup Hausdorff Markov and semigroup Fréchet Markov topology for $I_{\mathbb{N}}$.*

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The Polish inverse semigroup topology for $I_{\mathbb{N}}$

Topology I_4 on $I_{\mathbb{N}}$

I_4 is generated by $I_2 \cup I_3$ and has sub-basic sets $U_{m,n} = \{f \in I_{\mathbb{N}} : (m,n) \in f\}$, $W_m = \{f \in I_{\mathbb{N}} : m \notin \mathrm{dom}(f)\}$, and $W_m^{-1} = \{f \in I_{\mathbb{N}} : m \notin \mathrm{im}(f)\}$.

Theorem (L. Elliott, J. Jonušas, J.D. Mitchell, Z. Mesyan, M. Morayne, YP 2023)

The topology I_4 on $I_{\mathbb{N}}$ is

1. *a Polish inverse semigroup topology;*
2. *the inverse Hausdorff Markov, inverse Fréchet markov, and inverse Zariski topology;*
3. *the maximal second-countable semigroup topology;*
4. *the unique T_1 and second-countable inverse semigroup topology.*

Overview

$I_4 =$ generated by $I_2 \cup I_3$

$I_2 =$ generated by I_1 and $\{f \in I_{\mathbb{N}} : x \notin \mathrm{dom}(f)\}$

generated by I_1 and $\{f \in I_{\mathbb{N}} : x \notin \mathrm{im}(f)\}$ $= I_3$

$I_2 \cap I_3$

$I_1 =$ generated by the sets: $\{f \in I_{\mathbb{N}} : (x, y) \in f\}$, $\{f \in I_{\mathbb{N}} : (x, y) \notin f\}$

Are there any other Polish semigroup topologies for $I_{\mathbb{N}}$?

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Classifying Polish semigroup topologies on $I_{\mathbb{N}}$

Theorem (S. Bardyla, L. Elliott, J.D. Mitchell, YP 2024)

The semigroup $I_{\mathbb{N}}$ has countably infinitely many Polish semigroup topologies. The partial order of the Polish semigroup topologies on $I_{\mathbb{N}}$ contains every finite partial order, has infinite descending chains, but only finite ascending chains and anti-chains.

- The partial order of Polish semigroup topologies on $I_{\mathbb{N}}$ consists two dual intervals: $[I_2, I_4]$ and $[I_3, I_4]$.
- Topologies in $[I_2, I_4]$ are characterised by "waning" (in some sense decreasing) functions $f : \mathbb{N} \cup \{\infty\} \to \mathbb{N} \cup \{\infty\}$.
- For each waning function f, we get a Polish semigroup topology generated by the sets:

$$\{g \in I_{\mathbb{N}} : |\operatorname{im}(g) \setminus X| \geq n \text{ and } |X \cap \operatorname{im}(g)| \leq (n)f\}$$

over all $n \in \mathbb{N}$ and finite $X \subseteq \mathbb{N}$.

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Topologies on $I_{\mathbb{N}}$ and relational structures.

Recall the connection between the pointwise topology and relational structures:

- A submonoid of $\mathbb{N}^{\mathbb{N}}$ is closed if and only if it is the endomorphism monoid of a relational structure on $\mathbb{N}$.
- A subgroup of Sym($\mathbb{N}$) is closed if and only if it is the automorphism group of a relational structure on $\mathbb{N}$.

Theorem (M. Hampenberg, YP 2024)

Let M be an inverse submonoid of $I_{\mathbb{N}}$ which contains all idempotents of $I_{\mathbb{N}}$. Then the following are equivalent:

- *M is closed in some Polish semigroup topology on $I_{\mathbb{N}}$;*
- *M is closed in every shift-continuous T_1 topology on $I_{\mathbb{N}}$.*
- *M is the monoid of partial isomorphisms of a relational structure on $\mathbb{N}$;*

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Thank you for listening!